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CONTENTS

Bernoulli Numbers Modulo 27000	J. S. Frame	87
Rational Triangulations	Fred Supnick	95
New Loops from Old Geometries	J. M. Osborn	103
On Chromatic Graphs	Léopold Sauvé	107
The Wedge Product	Gerald Berman	112
The Complex Sum of Divisors	Robert Spira	120
Design of Mixed Doubles Tournaments	E. N. Gilbert	124
Mathematical Notes	H. L. Hunzeker, Martin Sandelius, David Zeitlin, W. H. Williams, W. E. Christilles, Ju-Kwei Wang, E. M. Wright, L. J. Mordell, W. A. Al-Salam	131
Classroom Notes	J. G. Christiano, O. J. Farrell and Bertram Ross, D. G. Mead, H. E. Stelson, Frank Harary, J. M. Gandhi, Edwin Halfar, J. W. Brown, Helen F. Cullen, F. H. Steen	149
Mathematical Education Notes	R. A. Rosenbaum	170
Elementary Problems and Solutions		177
Advanced Problems and Solutions		181
Recent Publications		188
News and Notices		199
The Mathematical Association of America		202
October Meeting of the Indiana Section		202
October Meeting of the Iowa Section		202
October Meeting of the Oklahoma Section		203
Calendar of Future Meetings		204

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The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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BERNOULLI NUMBERS MODULO 27000

J. S. FRAME, Michigan State University

1. Introduction. Bernoulli numbers play an important role in many interesting series expansions related to trigonometric and hyperbolic functions. They may be defined either as the coefficients B_k in the expansion

$$(1.1)\quad 1 - \tfrac{1}{2}x\cot\tfrac{1}{2}x = B(x) = \sum_1^\infty B_k\frac{x^{2k}}{(2k)!} = \frac{1}{6}\frac{x^2}{2!} + \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} + \cdots$$

or as the related coefficients b_k in the expansion

$$(1.2)\quad \frac{x}{e^x - 1} = b(x) = \sum_0^\infty b_k\frac{x^k}{k!} = 1 - \frac{x}{2} + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \cdots.$$

The function $b(x)+\frac{1}{2}x$ is equal to $\frac{1}{2}x\coth\frac{1}{2}x$ so the b_k are given in terms of B_k by the formulas

$$(1.3)\quad b_0 = 1,\ b_1 = -\tfrac{1}{2};\qquad b_{2k} = (-1)^{k-1}B_k,\ b_{2k+1} = 0 \qquad \text{for } k > 0.$$

Our purpose in this paper is to derive a number of congruence relations* for the B_k, using powers of 2, 3, and 5 and their products as moduli. Perhaps the most interesting of these relations is the congruence

$$(1.4)\qquad 30B_{2m} \equiv 1 + 600\binom{m-1}{2} \pmod{27000}.$$

2. Some important properties of Bernoulli numbers. From the identities

$$(2.1)\qquad x\tan\tfrac{1}{2}x = x\cot\tfrac{1}{2}x - 2x\cot x = 2[B(2x) - B(x)],$$

$$(2.2)\qquad x\csc x = x\cot\tfrac{1}{2}x - x\cot x = 1 + B(2x) - B(x),$$

it follows that the coefficients T_k in $x\tan\frac{1}{2}x$ and D_k in $x\csc x$ defined by

$$(2.3)\qquad x\tan\tfrac{1}{2}x = \sum_{k=1}^\infty T_k\frac{x^{2k}}{(2k)!},\qquad x\csc x = \sum_{k=0}^\infty D_k\frac{x^{2k}}{(2k)!},$$

are the following simple multiples of B_k:

$$(2.4)\qquad T_k = 2(4^k - 1)B_k,\qquad D_k = (4^k - 2)B_k = \tfrac{1}{2}T_k - B_k.$$

Since the coefficients T_k satisfy the recurrence relation [2]

$$(2.5)\quad T_1 = 1,\quad T_k = \binom{k}{2}T_{k-1} - \binom{k}{4}T_{k-2} + \binom{k}{6}T_{k-3} - \cdots,\ k > 1,$$

they are integers. Also since the coefficient of $x^{2k-1}/(2k-1)!$ in the expansion of $\tan x$ is the integer $4^{k-1}T_k/k$, it follows that T_k is an odd integer divisible by

* The author wishes to acknowledge the assistance of Mr. Ted Petrie in searching the literature on this subject.

the highest odd factor of k. On computing the first eight T_k from (2.5), and the corresponding B_k from (2.4), we find the following values:

(2.6)

k	1	2	3	4	5	6	7	8
T_k	1	1	3	17	155	2073	38227	929569
B_k	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$

It is clear from (2.4) that the denominator of B_k is twice an odd factor of 4^k-1. More precisely, this denominator is known by the Staudt-Clausen theorem [6] to be the product of all those distinct primes $p_{k,r}$ such that $p_{k,r}-1$ divides $2k$. In fact, Clausen and von Staudt [6] proved the congruence

$$(2.7) \qquad (-1)^k B_k \equiv \sum_r p_{k,r}^{-1} \pmod 1,$$

where $(p_{k,r}-1) \mid 2k$. For example

$$(2.8) \quad -B_1 = -\frac{1}{6} = \frac{1}{2}+\frac{1}{3}-1, \quad B_6 = \frac{691}{2730} = \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}-1.$$

Such equations imply congruences for the numerators N_k of B_k modulo each of the prime divisors of the denominator.

Besides these congruences for B_k with prime moduli, there are exact formulas for expressing B_k in terms of the sum of the series $\sum n^{-2k}$, and thus obtaining simple approximations to its magnitude. The function $(\sin \pi x)/(\pi x)$ vanishes at all the positive and negative integers and has the factorization

$$(2.9) \qquad \frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right).$$

Differentiation of the logarithm of both members gives

$$(2.10) \qquad \pi \cot \pi x - 1/x = -2x \sum_{n=1}^{\infty} (n^2 - x^2)^{-1}.$$

Comparing (2.10) and (1.1) we then see that

$$(2.11) \qquad B(2\pi x) = 2\sum_{n=1}^{\infty} \frac{x^2/n^2}{1 - x^2/n^2} = 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} x^{2k}/n^{2k}.$$

Then comparing coefficients of $2x^{2k}$ we find

$$(2.12) \qquad \frac{B_k}{2}\frac{(2\pi)^{2k}}{(2k)!} = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \zeta(2k) = \prod (1 - p^{-2k})^{-1},$$

where the last member of (2.12) is the factored form of the zeta function and the product is over all primes p. Use of Stirling's formula for $(2k)!$ yields the

approximation

$$B_k \sim 2\sqrt{(4\pi k)}(k/\pi e)^{2k}\zeta(2k) \sim \sqrt{(50k)}(0.117k)^{2k}\cdot 1 \quad \text{for } k > 4. \tag{2.13}$$

Also from (2.12) we may derive a close inequality for the ratio B_k/B_{k-1}:

$$1 - \frac{1}{4^{k-1}-2} < \frac{\pi^2}{k(k-\frac{1}{2})}\frac{B_k}{B_{k-1}} < 1 - \frac{3}{4^k-1}, \qquad k > 2. \tag{2.14}$$

The inequalities (2.14), together with the exact Staudt-Clausen formula for the denominator of B_k, are sufficient to determine the integer N_k if it is less than 4^k. For example, given $B_5 = 5/2\cdot 3\cdot 11$ and $B_6 = N_6/2\cdot 3\cdot 5\cdot 7\cdot 13$, (2.14) becomes for $k=6$

$$1 - \frac{1}{1022} < \frac{4\pi^2}{12\cdot 11}\,\frac{2\cdot 3\cdot 11}{2\cdot 3\cdot 5\cdot 7\cdot 13}\,\frac{N_6}{5} < 1 - \frac{1}{1365} \tag{2.15}$$

and we find

$$\frac{6825}{\pi^2}\cdot\frac{1021}{1022} = 690.840 < N_6 < \frac{6825}{\pi^2}\cdot\frac{1364}{1365} = 691.010. \tag{2.16}$$

Hence N_6 is the integer 691.

The exact computations in (2.16) can be avoided, however, if we use (1.4) together with a rough estimate of B_k. Since π^2 is nearly 10, we obtain from (2.13) or (2.14) the rough estimates $\frac{1}{4}$, 1, 7, 50, and 500 for $B_6, B_7, B_8, B_9, B_{10}$, and the corresponding rough estimates 680+, 6+, 3570+, 40000+, and 165,000 for their numerators. But by (1.4) we have

$$\begin{aligned} N_6 &= 91(30B_6) \equiv 91[1 + 600] \equiv 54691 \equiv 691 \pmod{27000}, \\ N_8 &= 17(30B_8) \equiv 17[1 + 1800] \equiv 30617 \equiv 3617 \pmod{27000}, \\ N_{10} &= 11(30B_{10}) \equiv 11[1 + 3600] \equiv 39611 \equiv 174611 \pmod{27000}. \end{aligned} \tag{2.17}$$

Thus the numerators N_6, N_8, and N_{10} are exactly 691, 3617, and 174611.

To derive the congruence (1.4) and a more complicated one with modulus 81000000 we first study congruences for B_k with moduli 4^9, 3^5, and 5^6.

3. Congruences modulo 4^9. We write the identity

$$\frac{2x}{e^x-1} - \frac{2x}{e^{2x}-1} = \frac{2xe^x}{e^{4x}-1} - \frac{2xe^{-x}}{e^{-4x}-1} \tag{3.1}$$

in the form

$$2b(x) - b(2x) = \tfrac{1}{2}[e^x b(4x) + e^{-x}b(-4x)]. \tag{3.2}$$

Then we equate coefficients of $x^{2n}/(2n)!$ using (2.1) and obtain

$$(2-4^n)b_n = 1 + 4b_1\binom{2n}{1} + 4^2 b_2\binom{2n}{2} + \cdots + 4^r b_r\binom{2n}{r} + \cdots. \tag{3.3}$$

Making use of the binomial identity

$$\binom{2n}{r} = \sum_k 2^{2k-r}\binom{k}{2k-r}\binom{n}{k} \tag{3.4}$$

and then replacing r by $2k-2s$ for even r, we obtain

$$\begin{aligned}(2-4^n)b_{2n} &= 1 - 4n + \sum_{k>s} 4^{2k-s}b_{2k-2s}\binom{k}{2s}\binom{n}{k} \\ &= 1 - 4n/3 + \sum_{k=2}^{\infty} 4^k\binom{n}{k}\sum_{s<k}\binom{k}{2s}4^{k-s}b_{2k-2s} \\ &= 1 - 4n/3 + \sum_{k=2}^{\infty} 4^k\binom{n}{k}\sum_{s<k}\binom{k}{2s}b_{2k-2s},\end{aligned} \tag{3.5}$$

where the last simplification is obtained by substituting T_k from (2.4) into (2.5). Thus B_n satisfies the congruence

$$\begin{aligned}(-1)^{n-1}(2-4^n)B_n \equiv 1 - \frac{4n}{3} + \frac{4^3}{30}\binom{n}{2} - \frac{4^5}{210}\binom{n}{3} + \frac{4^6}{210}\binom{n}{4} \\ - \frac{4^8}{462}\binom{n}{5} + \frac{191.4^9}{10010}\binom{n}{6} \pmod{2^{21}}.\end{aligned} \tag{3.6}$$

This was obtained by Frobenius by another method ([3], p. 821) with the comment that these results are hardly accessible by a derivation with the help of the exponential function.

In a recent paper [1], Carlitz stated that

$$2b_{2n} \equiv 1 \pmod{2^{r+1}}, \tag{3.7}$$

where 2^r is the highest power of 2 dividing $2n$. Clearly his congruence is a special case of (3.6).

For even n we set $n=2m$ in (3.6) and multiply by -15. Thus

$$\begin{aligned}(1-2^{4m-1})30B_{2m} \equiv -15 + 40m - 2^5\binom{2m}{2} + \frac{2^9}{7}\binom{2m}{3} \\ - \frac{2^{11}}{7}\binom{2m}{4} + \frac{5\cdot 2^{15}}{77}\binom{2m}{5} - 5\cdot 2^{17}\binom{2m}{6} \pmod{2^{21}}.\end{aligned} \tag{3.8}$$

Changing binomial coefficients to the exponent m we then have

$$\begin{aligned}&(1-2^{4m-1})30B_{2m} \\ &\equiv -15 + 8m - 2^7\binom{m}{2} - 3\cdot 2^{12}\binom{m}{3} + 9\cdot 2^{15}\binom{m}{4} \pmod{2^{19}}.\end{aligned} \tag{3.9}$$

Correcting for the term 2^{4m-1} for $m = 1, 2, 3, 4$ we have an expression for

$30B_{2m} \pmod{2^{18}}$.

$$30B_{2m} \equiv 1 + 2^3(m - 2 + \delta_{m1}) - 2^7\left[\binom{m}{2} - \delta_{m2}\right] - 2^{11}\left[6\binom{m}{3} - 9\delta_{m3}\right] + 2^{15}\left[9\binom{m}{4} + \delta_{m4}\right] \pmod{2^{18}}. \tag{3.10}$$

The corresponding formula for odd Bernoulli numbers is found to be

$$6B_{2m+1} \equiv (2\delta_{m0} - 1) + 8(4\delta_{m1} - 3)/7 + \frac{11 \cdot 2^7}{7}\binom{m}{2} - 2^9\delta_{m2} - 2^{12}\binom{m}{3} \pmod{2^{15}}. \tag{3.11}$$

4. Congruences with the modulus 3^5. We next write the identity

$$3b(x) - b(3x) = e^x b(3x) + e^{-x}b(-3x) \tag{4.1}$$

and equate coefficients of $x^{2n}/(2n)!$ on both sides. Thus

$$(3 - 3^{2n})b_{2n} = 2\sum_{r=0}^{2n}\binom{2n}{r}3^r b_r$$

$$\equiv 2 - 6n + 3\binom{2n}{2} - \frac{3^3}{5}\binom{2n}{4} \pmod{3^5} \tag{4.2}$$

$$\equiv 2 - 3n - 42\binom{n}{2} + 81\binom{n}{3} + 108\binom{n}{4} \pmod{3^5},$$

$$(-1)^n(1 - 3^{2n-1})30B_n$$

$$\equiv -20 + 30n - 66\binom{n}{2} - 81\binom{n}{3} + 108\binom{n}{4} \pmod{3^5}. \tag{4.3}$$

Using (3.4) again to change the binomial coefficients when $n = 2m$ is even we have

$$(1 - 3^{4m-1})30B_{2m} \equiv -20 - 6m + 33\binom{m}{2} - 27(64)\binom{m}{4} \pmod{3^5}. \tag{4.4}$$

Only when $m = 1$ does the term involving 3^{4m-1} appear, so

$$30B_{2m} \equiv 1 + 6\binom{m-1}{2} + 27\left[\delta_{m1} - 2 + m + \binom{m-1}{2} - 64\binom{m}{4}\right] \pmod{3^5}, \tag{4.5}$$

$$30B_{2m} \equiv 1 + 600\binom{m-1}{2} + 27\left[\delta_{m1} - 2 + m - 3\binom{m-1}{2} - 64\binom{m}{4}\right] \pmod{3^5}. \tag{4.6}$$

A corresponding congruence for the odd Bernoulli numbers is

$$6B_{2m+1} \equiv 3\delta_{m0} - 2 - 102m + 96\binom{m}{2} + 27\binom{m}{3} + 54\binom{m}{4} \pmod{3^5}. \tag{4.7}$$

5. Congruences with the modulus 5^6. Here we use the identity

$$5b(x) - b(5x) = [\cosh \tfrac{1}{2}x + \cosh \tfrac{3}{2}x]5x \operatorname{csch} \tfrac{5}{2}x. \tag{5.1}$$

Comparing coefficients of $x^{2n}/(2n)!$ and recalling (2.2), we have

$$\begin{aligned}(5 - 5^{2n})b_{2n} \equiv{}& 2\left[\left(\frac{5}{4} - 1\right)^n + \left(\frac{5}{4} + 1\right)^n\right]\binom{2n}{0} \\ &- \frac{2}{3}\left[\left(\frac{5}{4} - 1\right)^{n-1} + \left(\frac{5}{4} + 1\right)^{n-1}\right]\binom{2n}{2}\left(\frac{5}{2}\right)^2 \\ &+ \frac{14}{15}\left[\left(\frac{5}{4} - 1\right)^{n-2} + \left(\frac{5}{4} + 1\right)^{n-2}\right]\binom{2n}{4}\left(\frac{5}{2}\right)^4 \pmod{5^6}.\end{aligned} \tag{5.2}$$

Separate expansions must be used for odd and even n, and we consider only the case $n = 2m$, in expanding the binomials in 5/4.

$$\begin{aligned}(5 - 5^{2n})b_{2n} \equiv{}& 4 + 4\binom{n}{2}\left(\frac{5}{4}\right)^2 + 4\binom{n}{4}\left(\frac{5}{4}\right)^4 - \binom{2n}{3}\left(\frac{5}{2}\right)^3 \\ &- \frac{1}{6}\binom{n-1}{3}\binom{2n}{2}\left(\frac{5}{2}\right)^5 + \frac{14}{3}\binom{2n}{4}\left(\frac{5}{2}\right)^3 \pmod{5^6},\end{aligned} \tag{5.3}$$

$$\begin{aligned}(5 - 5^{2n})B_n \equiv{}& -4 - \binom{n}{2}\left(\frac{5}{2}\right)^2 + \binom{2n}{3}\left(\frac{5}{2}\right)^3 - \frac{5^4}{4^3}\binom{n}{4} \\ &+ \frac{2}{3}(2n-1)\binom{n}{4}\left(\frac{5}{2}\right)^5 - \frac{14}{3}\binom{2n}{4}\left(\frac{5}{2}\right)^3 \pmod{5^6}.\end{aligned} \tag{5.4}$$

Since $(2n+2)\binom{n}{4}$ is divisible by 5, the right member is a polynomial in n of degree 4 (mod 5^6). It has the simplified expansion

$$(5 - 5^{4m})B_{2m} \equiv -4 - \frac{2}{3}\binom{2m}{2}5^2 - 6\binom{2m}{3}5^3 + 4\binom{2m}{4}5^3 \pmod{5^6}. \tag{5.5}$$

We multiply by 6 and correct for the term in 5^{4m-1} when $m = 1$.

$$\begin{aligned}&(1 - 5^{4m-1})30B_{2m} \\ &\equiv 1 - 25 - 4\binom{2m}{2}5^2 - 36\binom{2m}{3}5^3 + 24\binom{2m}{4}5^3 \pmod{5^6} \\ &\equiv 1 - 25 - 100m + 225\binom{m}{2} + 384\binom{m}{4}5^3 \pmod{5^6},\end{aligned} \tag{5.6}$$

$$(5.7)\quad 30B_{2m} \equiv 1 + 600\binom{m-1}{2} + 5^3\left[+ m - 2 + \delta_{m1} - 3\binom{m-1}{2} + 384\binom{m}{4}\right] \pmod{5^6}.$$

We shall merely state without proof the corresponding congruence for the odd-numbered Bernoulli numbers, namely

$$(5.8)\quad 6B_{2m+1} \equiv 1 + 5(\delta_{m0} - 1) + 897m - 85\binom{m}{2} + 700\binom{m}{3} - 375\binom{m}{4} \pmod{5^5}.$$

6. Congruences modulo 27000 and 7776000000. We now combine the congruences for $30B_{2m}$ that we have found with modulus 2^{18} in (3.11), with modulus 3^5 in (4.6), and with modulus 5^6 in (5.7), noting that in all three cases the right member is a fourth-degree polynomial in m for $m>4$. Our main congruence (1.4) follows immediately, and we define the integers α, β, γ, ϵ and the rational number R with denominator prime to 30 by the abbreviations

$$(6.1)\quad \alpha = m - 2 + \delta_{m1},\quad \beta = \binom{m-1}{2},\quad \gamma = \binom{m}{4},\quad \epsilon = \binom{m}{2} - \delta_{m2},$$

$$(6.2)\quad 30B_{2m} = 1 + 600\beta + 27000R.$$

THEOREM. *The rational number R defined by* (6.2), *whose denominator is the same as that of $30B_{2m}$, satisfies the congruence*

$$(6.3)\quad R \equiv 11(\alpha - 3\beta) + 2500(5\alpha + 3\beta) + 224\gamma + 54000(\alpha - 3\beta - 2\gamma - \epsilon) \pmod{288000}.$$

(By substituting this value of R in (6.2), a congruence for $30B_{2m}$ is obtained modulo $2^{11}\cdot 3^5\cdot 5^6 = 7776000000$.)

Proof. Using the abbreviation (6.2) in (3.11), (4.6) and (5.7), and dividing by 2^3, 3^3 and 5^3, respectively, we have

$$(6.4)\quad (15)^3R \equiv \alpha - 75\beta - 16\epsilon \pmod{2^8},$$

$$(6.5)\quad (10)^3R \equiv \alpha - 3\beta - 64\gamma \pmod{3^2},$$

$$(6.6)\quad 6^3R \equiv \alpha - 3\beta + 9\gamma \pmod{5^3}.$$

Since $(1+5)^{-3} \equiv 11 \pmod{5^3}$, we satisfy (6.6) by introducing a fraction S with denominator prime to 5 such that

$$(6.7)\quad R = 11(\alpha - 3\beta + 9\gamma) + 125(\gamma + S).$$

The congruence (6.5) is now satisfied by introducing T so that

$$(6.8)\quad S = 100\alpha + 60\beta + 9T.$$

To simplify (6.4) we multiply both sides by -49, which is congruent to

$(-1+16)^{-3} \pmod{2^8}$. Substitution of the value of R from (6.7) in (6.4) then gives

$$(6.9) \qquad 11(\alpha - 3\beta) + 224\gamma + 125S \equiv -49\alpha + 49(75)\beta + 16\epsilon \pmod{2^8}.$$

Solving for $125S$ and using (6.8) we obtain

$$(6.10) \qquad \begin{aligned} 125(100\alpha + 60\beta + 9T) &\equiv -60\alpha + 124\beta + 32\gamma + 16\epsilon \pmod{2^8}, \\ -3(25\alpha + 15\beta + 9T/4) &\equiv -15\alpha + 31\beta + 8\gamma + 4\epsilon \pmod{2^6}. \end{aligned}$$

Solving for $T/4$ we find

$$(6.11) \qquad T/4 \equiv 12(\alpha - 3\beta - 2\gamma - \epsilon) \pmod{2^6}.$$

The congruence (6.3) now follows from (6.7), (6.8) and (6.11).

Although it is less elegant than the congruence for B_{2m}, we include an analogous congruence for B_{2m+1} derived from (3.13), (4.7) and (5.8):

$$(6.12) \qquad \begin{aligned} 6B_{2m+1} \equiv 1 - 6m/7 &+ 2040\binom{m}{2} - 60\binom{m}{3} \\ &+ 8130\left[\delta_{m0} + \binom{m-1}{3} + \binom{m+1}{3}\right] \pmod{27000}. \end{aligned}$$

In conclusion we compute $B_{12} = N_{12}/2730$ by (6.2) and (6.3), using (2.13) to obtain the approximate magnitude

$$(6.13) \qquad N_{12} \sim \sqrt{(600)}(1.404)^{24}.2730 \sim 2.3 \times 10^8.$$

For $m = 6$, (6.1) gives the parameter values

$$(6.14) \qquad \alpha = 4, \quad \beta = 10, \quad \gamma = 15, \quad \epsilon = 15,$$

and (6.2) gives

$$(6.15) \qquad N_{12} = 91(30B_{12}) = 546091 + 27000(91R).$$

From (6.13) and (6.15) the magnitude of $91R$ is about 9000. From the congruence in our main theorem we have

$$(6.16) \qquad \begin{aligned} 91R &\equiv 1001(-26) + 91(2500)(50) + 91(3360) \pmod{18000} \\ &\equiv 11654734 \equiv 8734 \pmod{18000}. \end{aligned}$$

Hence $91R$ is 8734 and

$$(6.17) \qquad N_{12} = 546091 + 27000(8734) = 236364091.$$

The twelfth Bernoulli number is $B_{12} = 236364091/2730$.

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RATIONAL TRIANGULATIONS

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1. Introduction. A triangle with rational sides and rational area is referred to as a *rational triangle.* Let S_1 denote the configuration consisting of three mutually tangent circles which are not tangent at one point (some but not all three circles may be straight lines; if two are straight lines then they must be parallel). Let S_i denote the configuration consisting of S_{i-1} and all circles each of which is tangent to three mutually tangent circles of S_{i-1}. Let S denote the smallest configuration containing each S_i $(i=1, 2, \cdots)$. Let w be a triangle whose vertices are the centers of three mutually tangent circles of finite radius of S. Let W denote the set of all triangles w.

We prove the following theorem:

THEOREM 1. *Either all or none of the triangles of W are rational.*

The theorem can be applied to obtain a class of solutions of the functional equation

$$f(x_1^*)f(x_2^*)f(x_3^*)(f(x_1^*)+f(x_2^*)+f(x_3^*))=R^2, \tag{1.1}$$

R rational (*i.e.*, functions $f(x^*)$ and suitable triplets (x_1^*, x_2^*, x_3^*) can be determined satisfying (1.1)). Suppose that the triangles of W were rational. If it were possible to obtain a function $f(x^*)$ defined over a domain D, and triplets $\{x_{1,i}^*, x_{2,i}^*, x_{3,i}^*\} \subset D$ $(i=1, 2, \cdots)$, so that $f(x_{1,i}^*)$, $f(x_{2,i}^*)$, $f(x_{3,i}^*)$ were the lengths of the radii of mutually tangent circles $C(x_{1,i}^*)$, $C(x_{2,i}^*)$, $C(x_{3,i}^*)$ of S with no interior point in common, then the area

$$(f(x_{1,i}^*)f(x_{2,i}^*)f(x_{3,i}^*)(f(x_{1,i}^*)+f(x_{2,i}^*)+f(x_{3,i}^*)))^{1/2}$$

(by Heron's Formula) of the triangle having the centers of $C(x_{1,i}^*)$, $C(x_{2,i}^*)$, $C(x_{3,i}^*)$ as vertices would be rational. In Section 3 we show how such functions $f(x^*)$ and triplets (x_1^*, x_2^*, x_3^*) may indeed be obtained. Namely, by executing a geometric inversion on the circle configuration attributed to A. Speiser (cf. [1]; also [2]) into a configuration S whose circles have no interior points in common, and whose triangles W are all rational. The formula for the radius of the inverse of a given circle gives a suitable $f(x^*)$, and those consecutive triplets (α, β, γ) of any Farey series F_j (cf. Sec. 3 below) for which $\beta \notin F_{j-1}$ gives a triplet (x_1^*, x_2^*, x_3^*).

By this method, we may obtain classes of (integral or rational) solutions of the Diophantine equation

$$xy + yz + xz = R^2 \tag{1.2}$$

(cf. Remark 1 below). We shall make use of the following theorem of H. E. Blichfeldt [3]:

If the sides of a triangle are a, b, c, then by letting

$$a = \left| m + \frac{1}{m} \right|, \qquad b = \left| n + \frac{1}{n} \right|, \qquad c = \left| m - \frac{1}{m} + n - \frac{1}{n} \right|,$$

and giving m and n all rational values integral or fractional, positive or negative, we form all the triangles with rational sides and having rational areas.

We illustrate the method outlined above by deriving those solutions of the Diophantine equation (1.2) which are associated with some one, say the (+, +, +) case, of the eight possibilities occurring in Blichfeldt's theorem:

$$\begin{aligned} 0 < a &= \pm\left(m + \frac{1}{m}\right), \qquad 0 < b = \pm\left(n + \frac{1}{n}\right), \\ 0 < c &= \pm\left(m - \frac{1}{m} + n - \frac{1}{n}\right), \end{aligned} \tag{1.3}$$

namely, the case $m, n > 0$, $mn > 1$ (cf. Remark 2 below) which is expressed by the following theorem:

THEOREM 2. *Let p_1/q_1, p_2/q_2, p_3/q_3 be consecutive elements of a Farey series, F_j, such that $p_2/q_2 \notin F_{j-1}$. Let*

$$\begin{aligned} G(m, n, p, q) \equiv\ & q^2n^2(m^2 + 1) + 2mn(m + n)q(q - p) \\ & + m^2(n^2 + 1)(q - p)^2 - mn(mn - 1), \end{aligned} \tag{1.4}$$

$G_1 \equiv G(m, n, p_1, q_1)$, $G_2 \equiv G(m, n, p_2, q_2)$, $G_3 \equiv G(m, n, p_3, q_3)$. If $m, n \geqq 0$, $mn \geqq 1$, then

$$G_1G_2 + G_1G_3 + G_2G_3 = R^2, \tag{1.5}$$

where R is integral or rational accordingly as both m and n are integral or rational.

Remark 1. We note that (1.2) is a generalization of the "Pell" equation

$$3t^2 - R^2 = 1, \tag{1.6}$$

since, if we let

$$x = t, \qquad y = t - 1, \qquad z = t + 1 \tag{1.7}$$

in (1.2), we obtain (1.6). Thus any solution of (1.6) provides through (1.7) a solution of (1.2).

Remark 2. By the (+, +, +) case we mean, of course, the choice of the "+" signs in the right expressions of (1.3). Noticing that

$$m - \frac{1}{m} + n - \frac{1}{n} = (m+n)\left(\frac{mn-1}{mn}\right),$$

and that a, b, c must be positive, we see that the (+, +, +) case is equivalent to $m, n>0$, $mn>1$. For the (+, +, −) case we have

$$\left(m+\frac{1}{m}\right)>0, \quad \left(n+\frac{1}{n}\right)>0, \quad -\left(m-\frac{1}{m}+n-\frac{1}{n}\right)>0,$$

which is equivalent to $m, n>0$, $mn<1$; etc.

The method of obtaining solutions of (1.2) in the (+, +, +) case, illustrated in the proof of Theorem 2, may be applied to the other cases (+, +, −), (+, −, +), (+, −, −), (−, +, +), (−, +, −), (−, −, +), (−, −, −) of (1.3), each leading to a class of solutions of (1.2).

2. Some lemmas and proof of the theorem. We prove

LEMMA 1. *Let C_x, C_y, C_z, C_t be mutually tangent circles (i.e. each tangent to the other three) with radii of finite lengths x, y, z, t and centers X, Y, Z, T respectively. If ΔXYZ is rational, then*

$$\Delta YZT, \ \Delta XYT, \ \Delta XZT \tag{2.1}$$

are also rational.

Proof. I. Suppose C_x, C_y, C_z, C_t are mutually external (*i.e.*, have no interior point in common). Since the sides of ΔXYZ are rational, x, y, z must be rational. By a geometric inversion (which leaves C_x invariant, with center at the point of tangency between C_y and C_z), it may be shown that

$$t = \frac{xyz}{x(y+z) \pm 2A + yz}, \tag{2.2}$$

where $A \equiv (xyz(x+y+z))^{\frac{1}{2}}$ is the area of ΔXYZ, and upper or lower signs are to be used accordingly as T is inside or outside ΔXYZ (cf. note 1 below). Since A is rational, t is rational. Thus the sides of triangles (2.1) are rational. The areas

$$(tyz(t+y+z))^{1/2}, \quad (txy(t+x+y))^{1/2}, \quad (txz(t+x+z))^{1/2}$$

of triangles (2.1) are

$$t(y+z \pm (A/x)), \quad t(x+y \pm (A/z)), \quad t(x+z \pm (A/y))$$

respectively (cf. Note 2 below), and are therefore rational (upper or lower signs taken as before).

II. Suppose C_x, C_y, C_z are mutually external, but are all inside C_t. Then

$$t = \frac{xyz}{-(x(y+z) - 2A + yz)}. \tag{2.3}$$

Thus the sides of triangles (2.1) are rational. The areas are also rational; thus, *e.g.*, the area of ΔTYZ is $(tyz(t-y-z))^{\frac{1}{2}}$, which is equal to $t(y+z-(A/x))$.

III. Suppose C_t and two elements C_g, C_h of $\{C_x, C_y, C_z\}$ are mutually external, and C_t, C_g, C_h are all inside $C_k (C_k \in \{C_x, C_y, C_z\}, C_k \neq C_g, C_k \neq C_h)$. Thus, *e.g.*, suppose C_y, C_z, C_t are mutually external and are all inside C_x. The area A' of ΔXYZ is $(xyz(x-y-z))^{\frac{1}{2}}$ (which is rational by hypothesis). Now,

$$t = \left| \frac{xyz}{x(y+z) \pm 2A' - yz} \right|. \tag{2.4}$$

If $y+z=x$, then $A'=0$. If $y+z \neq x$, then there are two possible circles C_t (of unequal radii); the $+$ or $-$ is taken accordingly as C_t is the smaller or larger of these possibilities. The sides of triangles (2.1) are of course rational. The areas

$$(tyz(t+y+z))^{1/2}, \qquad (txy(x-y-t))^{1/2}, \qquad (txz(x-z-t))^{1/2}$$

of triangles (2.1) are

$$t(y+z \pm (A'/x)), \qquad t(x-y \pm (A'/z)), \qquad t(x-z \pm (A'/y))$$

respectively, and are therefore rational.

Note 1. Proof of the "$+$" case of (2.2) (the "$-$" case, (2.3), (2.4), (2.5), (2.6) and (2.7) are done in an analogous manner): Suppose C_x, C_y, C_z, C_t are mutually and *externally* tangent (*i.e.*, they have no interior point in common) and T is inside ΔXYZ. Let I be the inversion which leaves C_x invariant and whose center is at the point P of tangency between C_y and C_z. Then $I(C_y)$ (the inverse of C_y) and $I(C_z)$ are lines which we denote by l_y and l_z respectively. Let l_y and l_z intersect the line $l(Y, Z)$ (determined by Y and Z) in the points L_y and L_z, respectively. Let k denote the length of the radius of the circle of inversion. Then $k^2 = \overline{PL_y} \cdot 2y$, $k^2 = \overline{PL_z} \cdot 2z$ and $\overline{PL_y} + \overline{PL_z} = 2x$. Eliminating $\overline{PL_y}$ and $\overline{PL_z}$ we obtain $k^2 = 4xyz/(y+z)$. Let D_t denote $I(C_t)$, and d_t be the center of D_t; the radius of D_t is x. Let h be the distance from X to $l(Y, Z)$. Now, P is an external homothetic center between C_t and D_t. Let one of the common tangents to C_t and D_t which pass through P touch them in the points A and B, respectively. Suppose (with no loss of generality) that $z \geqq y$. Then, since

$$h = 2\{xyz(x+y+z)\}^{1/2}/(y+z), \qquad \overline{PL_x} = 2xy/(y+z),$$

we obtain

$$t = x \cdot \frac{\overline{PA}}{\overline{PB}} = x \cdot \frac{\overline{PA} \cdot \overline{PB}}{\overline{PB}^2} = \frac{x \cdot k^2}{\overline{Pd_t}^2 - x^2}$$

$$= \frac{x \cdot k^2}{(x - \overline{PL_x})^2 + (h+2x)^2 - x^2} = \frac{xyz}{x(y+z) + 2\{xyz(x+y+z)\}^{1/2} + yz}.$$

Note 2. Thus, let K denote the denoninator of (2.2). Then

$$[tyz(t+y+z)]^{1/2} = \left[\frac{xyz}{K}\cdot yz\left(\frac{xyz}{K}+y+z\right)\right]^{1/2}$$

$$= \frac{yz}{K}\left[x\{xyz+(y+z)K\}\right]^{1/2} = \frac{yz}{K}\left[\{x(y+z)\pm[xyz(x+y+z)]^{1/2}\}^2\right]^{1/2}$$

$$= \frac{yz}{K}(x(y+z)\pm A) = t\left(y+z\pm\frac{A}{x}\right).$$

LEMMA 2. *Let mutually and externally tangent circles C_x, C_y, C_z with centers at X, Y, Z, and radii of rational lengths x, y, z respectively $(x<\infty, y<\infty, z<\min(x,y))$, be tangent to a line L. Let C_t be a circle tangent to L and any two elements C_g, C_h of $\{C_x, C_y, C_z\}$; let G, H, T be the centers, and $g, h, t(t<\infty)$ be the lengths of the radii of C_g, C_h, C_t respectively. Then ΔGHT is rational.*

Proof. Suppose $C_t\in\{C_x, C_y, C_z\}$. Then $\Delta XYZ\equiv\Delta GHT$. We note that ΔXYZ is rational. Its sides are of course rational. Since

$$z = \frac{xy}{x+y+2(xy)^{1/2}}, \tag{2.5}$$

then $(xy)^{\frac{1}{2}}$ is rational. But the area $(xyz(x+y+z))^{\frac{1}{2}}$ of ΔXYZ is also rational since it is equal to $z(x+y+(xy)^{\frac{1}{2}})$. Let us therefore assume that $C_t\notin\{C_x, C_y, C_z\}$.

Suppose $C_g\equiv C_y$, $C_h\equiv C_z$. Let x_{-1} be the radius of the circle $C_{x_{-1}}(\not\equiv C_z)$ which is tangent to C_x, C_y and L. Let x_2 be the radius of the circle $C_{x_2}(\not\equiv C_x)$ which is tangent to C_z, C_y and L. Then

$$x_{-1} = \frac{xy}{x+y-2(xy)^{1/2}}, \tag{2.6}$$

$$x_2 = \frac{xy}{4x+y+4(xy)^{1/2}}. \tag{2.7}$$

Since $(xy)^{\frac{1}{2}}$ is rational (cf. (2.5)) x_{-1} and x_2 are rational.

Suppose $C_t\equiv X_{x_2}$. The sides of ΔTYZ are rational. The area $(zx_2y(z+x_2+y))^{\frac{1}{2}}$ of ΔTYZ (using (2.5) and (2.7)) is $zx_2(3x+3(xy)^{\frac{1}{2}}+y)/x$, and is therefore rational.

Similarly, if $C_g\equiv C_x$, $C_h\equiv C_z$ we can show that ΔTZX is rational.

Suppose $C_g\equiv C_y$, $C_h\equiv C_x$. There is no loss of generality if we assume $x<y$ (if $x=y$, then $t=\infty$ which is contrary to hypothesis). Let $C_t=C_{x_{-1}}$. The sides of ΔXYT are rational. The area $(x_{-1}xy(x_{-1}+x+y))^{\frac{1}{2}}$ of ΔXYT is $x_{-1}(x+y-(xy)^{\frac{1}{2}})$ and is therefore rational. This completes the proof of Lemma 2.

Proof of the theorem. There is no loss of generality if we assume that S_1 consists of mutually tangent circles (not all tangent at one point) with centers at X, Y, Z $(x<\infty, y<\infty, z<\infty)$. Let us assume that ΔXYZ is rational. Let w_i

denote the set of all (nondegenerate) triangles whose vertices are three mutually tangent circles of S_i.

If no S_i has a straight line as one of its "circles," then by Lemma 1 each element of w_i $(i=1, 2, \cdots)$ is rational.

Suppose S_{i_0} does not have a straight line as one of its "circles," and S_{i_0+1} does. Then by Lemma 1 each element of w_{i_0} is rational, and by Lemmas 1 and 2 each element of w_i $(i>i_0)$ is rational.

COROLLARY. *If any four mutually tangent circles of S have radii of (finite) rational length, then all elements of W are rational.*

Proof. This is an immediate consequence of (2.2), (2.3), (2.4) and the theorem.

3. A class of solutions of (1.1) and proof of Theorem 2. Let F_j denote the Farey series of order j, *i.e.*, the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed j, the numbers 0 and 1 included in the form 0/1 and 1/1 (cf. [4]).

Consider the circle $C_{p/q}$ with center at $(p/q, 1/2q^2)$ (cartesian coordinates) and radius $1/2q^2$, where p and q are relatively prime integers, and $0 \leqq p/q \leqq 1$. Let S' denote the configuration consisting of all such circles $C_{p/q}$ and no others (cf. [1], [2]). If p_1/q_1, p_2/q_2, p_3/q_3 are consecutive terms of F_j for which $p_2/q_2 \notin F_{j-1}$, then C_{p_i/q_i} $(i=1, 2, 3)$ are mutually and externally tangent.

The triangle with vertices $(0, \frac{1}{2})$, $(1, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{8})$ is rational. Therefore any triangle whose vertices are the centers of three mutually tangent circles of S' are *rational* triangles. The area of the triangle with vertices $(p_i/q_i, 1/2q_i^2)$ $(i=1, 2, 3)$ is therefore rational, *i.e.*

$$\left\{\frac{1}{2q_1^2}\cdot\frac{1}{2q_2^2}\cdot\frac{1}{2q_3^2}\left(\frac{1}{2q_1^2}+\frac{1}{2q_2^2}+\frac{1}{2q_3^2}\right)\right\}^{1/2}$$

is rational. Thus the function $f(x^*)=1/2q^2 (x^*=p/q, (p, q)=1)$ satisfies (1.1) for all triplets $x_1^* \equiv p_1/q_1$, $x_2^* \equiv p_2/q_2$, $x_3^* \equiv p_3/q_3$.

Now, let I be an inversion which takes any three mutually tangent circles of S' into mutually external circles whose centers are vertices of a rational triangle. Let $f(p/q)$ be the length of the radius of $I(C_{p/q})$. Then

$$f(p_1/q_1)f(p_2/q_2)f(p_3/q_3)[f(p_1/q_1)+f(p_2/q_2)+f(p_3/q_3)] = R^2, \tag{3.1}$$

R rational. We now show how to construct such an inversion I, and obtain the corresponding function $f(p/q)$ explicitly.

Let m and n be positive rational numbers for which $mn>1$. Let a triangle τ have sides a, b, c with

$$a = m + m^{-1}, \qquad b = n + n^{-1}, \qquad c = m - m^{-1} + n - n^{-1}.$$

Let $a=x+y$, $b=y+z$, $c=x+z$. Then

$$x = m - n^{-1}, \qquad y = m^{-1} + n^{-1}, \qquad z = n - m^{-1}. \tag{3.2}$$

Consider mutually and externally tangent circles C_x, C_y, C_z with radii of lengths x, y, z (cf. (3.2)) and centers at X, Y, Z the common endpoints of sides a and c, a and b, b and c, respectively. Let D be a circle, with diameter of length one and center at J, which is externally tangent to C_y and C_z, where J and X are in the same half-plane determined by the line $l(Y, Z)$.

Let us now consider the inversion I which leaves D invariant and whose center is at the point of tangency between C_y and C_z. $I(C_y)$ and $I(C_z)$ are lines which we denote by l_y and l_z respectively. Let $l(Y, Z)$ intersect l_y and l_z in the points L_y and L_z respectively. Let C_t be the circle tangent to but not overlapping C_x, C_y, C_z. Let E and F be the points of tangency between l_y and the inverses $I(C_t)$ and $I(C_z)$ of C_t and C_z respectively. Let l_E be the line perpendicular to l_y at E. Let l_y and l_E be the axes of a cartesian coordinate system with E: (0, 0) and F: (1, 0). Let us consider the circle configuration S' on this coordinate system, and let $C_{p/q}$ be defined as above, its center being denoted by G.

We note that P is an external homothetic center between $C_{p/q}$ and $I(C_{p/q})$. Let one of the common tangents to $C_{p/q}$ and $I(C_{p/q})$ which pass through P, touch $C_{p/q}$ and $I(C_{p/q})$ in the points B and A, respectively.

There is no loss in generality if we assume $z \geqq y$. Since I leaves D invariant, the power k of I satisfies

$$k^2 = \overline{PL}_y \cdot 2y, \qquad k^2 = \overline{PL}_z \cdot 2z, \qquad \overline{PL}_y + \overline{PL}_z = 1. \tag{3.3}$$

Eliminating $\overline{PL}_y$ and $\overline{PL}_z$ we obtain

$$k^2 = 2yz/(y + z). \tag{3.4}$$

Furthermore, using (3.4) in the first expression of (3.3) we obtain

$$\overline{PL}_y = z/(y + z). \tag{3.5}$$

Let h denote the distance from X to $l(Y, Z)$. Then

$$h = 2(xyz(x + y + z))^{1/2}/(y + z). \tag{3.6}$$

$I(C_z)$ has center at X': $(1, \frac{1}{2})$. Let H be the distance from X' to $l(Y, Z)$. Then $H/h = \overline{X'P}/\overline{XP} = \frac{1}{2}/x$, *i.e.*,

$$H = (xyz(x + y + z))^{1/2}/x(y + z). \tag{3.7}$$

The radius $f(p/q)$ of $I(C_{p/q})$ may now be computed as follows:

$$f(p/q) = \frac{1}{2q^2}\,\frac{\overline{PA}}{\overline{PB}} = \frac{1}{2q^2}\,\frac{\overline{PA}\cdot\overline{PB}}{\overline{PB}^2} = \frac{1}{2q^2}\,\frac{k^2}{\overline{PG}^2 - \{1/(2q^2)\}^2}.$$

Thus

$$f(p/q) = \frac{\{1/(2\cdot q^2)\}k^2}{(\overline{PL}_y - \{(1/(2\cdot q^2)\})^2 + (H + 1 - (p/q))^2 - \{1/(2\cdot q^2)\}^2}. \tag{3.8}$$

Applying (3.4), (3.5), (3.7) to (3.8), and then (3.2), we obtain finally

$$(3.9)\quad f(p/q)=\frac{(m+n)(mn-1)}{q^2n^2(m^2+1)+2mn(m+n)q(q-p)+m^2(n^2+1)(q-p)^2-mn(mn-1)}.$$

We recall that

$$(3.10)\qquad C_{p_1/q_1},\ C_{p_2/q_2},\ C_{p_3/q_3}$$

are mutually and externally tangent since p_1/q_1, p_2/q_2, p_3/q_3 are consecutive terms of F_j such that $p_2/q_2 \notin F_{j-1}$. Since the center of inversion P is not inside any circle of (3.10), the inverses of (3.10),

$$(3.11)\qquad I(C_{p_1/q_1}),\ I(C_{p_2/q_2}),\ I(C_{p_3/q_3}),$$

must be mutually and *externally* tangent. As the triangle τ with sides a, b, c is rational (by Blichfieldt's theorem), the triangle joining the centers of (3.11) is rational (by Th. 1). Thus $f(p/q)$ satisfies (1.1) for all consecutive triplets $(p_1/q_1,\ p_2/q_2,\ p_3/q_3)$ of any Farey series F_j for which $p_2/q_2 \notin F_{j-1}$.

We now complete the proof of Theorem 2. Equation (3.1) is valid for $f(p/q)$ defined by (3.9) over the set Γ of all pairs of positive rational numbers (m, n) for which $mn>1$. Substituting (3.9) into (3.1) we see that (1.5) is valid over Γ. We note now, that if m and n are integers, the left side of (1.5) is an integer, and this implies that R is integral.

Finally, we note that (1.5) holds for $G(0, n, p, q)$, $G(m, 0, p, q)$, or where $mn=1$. This is a consequence of the identity

$$(M^2 + MN + N^2)^2 = (MN)^2 + \{M(M + N)\}^2 + \{N(M + N)\}^2$$

(cf. [5]), and the fact that $p_2=p_1+p_3$ and $q_2=q_1+q_3$ (since $p_2/q_2 \notin F_{j-1}$).

Remark 3. In the above argument, the center of inversion P was the point of contact between C_y and C_z. The cases where the inversion center P is at the contact point between C_z and C_x, or between C_x and C_y, is treated analogously. There are, in general, three positions for P for each of the eight cases of (1.3).

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NEW LOOPS FROM OLD GEOMETRIES

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In this paper we shall exhibit a simple but unusual way of defining an algebraic system on the points of a Euclidean plane. Although our ultimate purpose is to produce examples of certain algebraic systems the theory of which has been treated elsewhere,* we shall try to make this paper as self-contained as possible, since we expect the method to be of wider interest than the resulting examples.

We begin by defining a new binary operation on the complex numbers. Let a be a fixed complex number not zero or one, and define

$$x \circ y = x + a(y - x) = (1 - a)x + ay \tag{1}$$

for any two complex numbers x and y. From this definition it is easy to check that

(i) If any two of the three complex numbers x, y, z are chosen, there exists a unique choice for the third one so that the relation $x \circ y = z$ is satisfied,

(ii) For any four complex numbers w, x, y, and z, $(w \circ x) \circ (y \circ z) = (w \circ y) \circ (x \circ z)$,

(iii) For any complex number x, $x \circ x = x$.

A set with a binary operation satisfying these three postulates is called an *idempotent Abelian quasigroup*. Thus, for each choice of the complex number a, we have made the set of complex numbers into an idempotent Abelian quasigroup, which we shall denote by Q.

Let us next think of the complex numbers as the points of a plane E, and consider what happens geometrically under the operation "$\circ$." If $a = re^{-i\theta}$ for real numbers $r > 0$ and $0 \leqq \theta < 2\pi$, then (1) may be expressed geometrically by saying that $z = x \circ y$ is that point of the plane such that the directed angle between the segments $\overline{xz}$ and $\overline{xy}$ is θ, and such that the length of the segment $\overline{xz}$ is r times the length of the segment $\overline{xy}$. But from this geometric description of Q it is clear that any translation of the plane induces an automorphism on Q. This can be expressed algebraically by the identity

$$x + (y \circ z) = (x + y) \circ (x + z), \tag{2}$$

where "$+$" is the usual operation of addition of complex numbers. We thus have two operations ("$+$" and "$\circ$") defined on the set of complex numbers (or on the points of the plane E), one of which distributes over the other—a situation which is reminiscent of a field. It will prove worth while here to alter our system for the purpose of strengthening this resemblance.

Let L be the set consisting of the symbol u and the points of E under a new operation "$\circ$" which is exactly the same as the old operation "$\circ$" for the product

* Specifically, we desire examples of cross inverse property loops, which have been discussed in [1], [2] and [7].

of any two distinct points of E, and which is defined otherwise by $x \circ u = u \circ x = x$ and $x \circ x = u$. Now L still satisfies (i), and in addition has a two-sided identity element u. Such a system is called a *loop*. Making the obvious definitions $x+u=u+x=u+u=u$, the distributive law (2) still holds, giving an algebraic system which now only fails in satisfying the axioms of a field because the operation "$\circ$" is not associative. A set with two binary operations satisfying all the axioms of a field except for the associative law for one or both of these operations is called a *neofield*.* We may then summarize the above remarks by saying that we have exhibited a class of neofields.

Let us next restrict our attention to the special case of the loop L mentioned above in which r and θ have the values 1 and $\frac{1}{3}\pi$, respectively. Geometrically, this is the case where the points x, y and $x \circ y$ form the vertices of an equilateral triangle. Rotating this triangle, we may then conclude that

$$(x \circ y) \circ x = y, \tag{3}$$

a relation which still holds if either x or y is the identity u. It is examples of loops satisfying this relation (known as the cross inverse property†) that we wish to derive. If we ask for the subloop K of L generated by two distinct points of E, we will find that we get exactly the vertices of a regular triangular tessellation of the plane‡ (illustrated in Fig. 1) in addition to u. And if we ask for the subloop K_1 of K generated by two vertices of this tessellation which are not adjacent, we get a proper subset of the vertices which themselves are the vertices of a regular triangular tessellation (for example, the set of vertices which are labelled "1" in Fig. 1). Considering the translates of the tessellation K_1^* corresponding to K_1 within the tessellation K^* corresponding to K, it is clear that K^* may be broken up into a finite number of equivalence classes, each of which is a tessellation. But since the subloops of L are just the subquasigroups of Q with the identity adjoined, and since it is known that the cosets of an Abelian quasigroup with respect to any subquasigroup also form a quasigroup,§ we may conclude that the cosets of K modulo K_1 form a finite homomorph H of K. For example, identifying together the vertices in Figure 1 which have the same label, we get a quasigroup of order 7, and adding an identity element gives a loop of order 8. As in the case of the whole plane, we may identify each translation of the tessellation K^* with an element of H to make H into a neofield.

We shall determine next the orders of the loops arising from the homomorphs of K. Given two vertices of K_1^* which are a minimal distance apart, we observe that there is a unique parallelogram P in K^* with these two points as the

* A more precise definition of a neofield may be found in [8].

† In the usual definition of this property, one of the x's in (3) is replaced by an inverse of x; but since we have defined $xox=u$, x and its inverses are the same here.

‡ A tessellation is an infinite collection of polygons which cover the whole plane without overlapping, and which have the property that every edge of each polygon belongs also to one other polygon.

§ See [5].

acute-angled vertices of P. Let the minimal distance between vertices in K^* be one unit, and let the lengths of two nonparallel sides of this parallelogram be m and n (in Fig. 1, m and n have the values 2 and 1, respectively, for example). Then the minimal distance between vertices in K_1^* is just the length of the longer diagonal of P, which is easily computed to be $t=\sqrt{(m^2+mn+n^2)}$. But the number of disjoint translates of K_1^* in K^* is the same as the ratio of the density of the vertices of K^* in the plane to the density of the vertices of K_1^* in the plane. And this is the same as the square of the ratio of the minimal distances occurring between vertices in K_1^* and K^* respectively, or $t^2=m^2+mn+n^2$. Since we can find a subtessellation of K^* using any nonnegative integral values of m and n we wish (m, n not both zero), we have shown that the orders of the loops arising by adjoining an identity element to each of the proper homomorphs of K, are exactly the integers of the form m^2+mn+n^2+1.

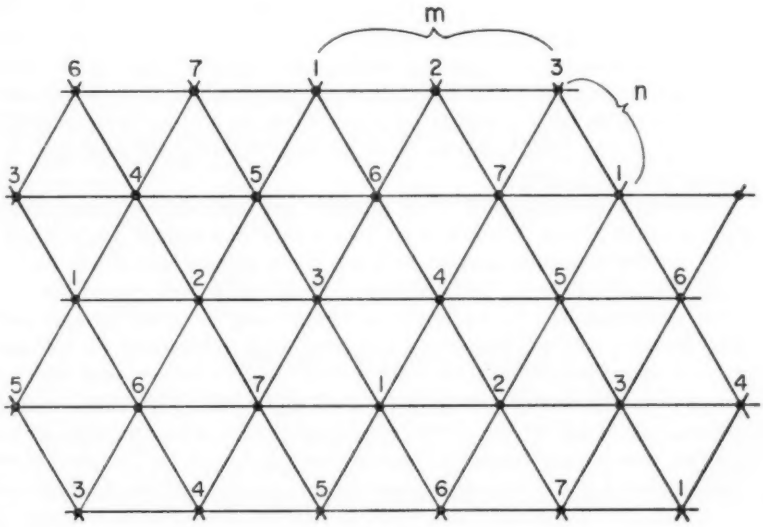

FIG. 1

The orders occurring here may be characterized a little more neatly using the following result which we quote from number theory.*

LEMMA. *A positive integer may be expressed in the form m^2+mn+n^2, where m and n are nonnegative integers, if and only if its square-free part is not divisible by a prime of the form $6t+5$.*

We are now ready to state formally what we have proved.

* For example, see Chapter VI and page 265 of [6].

THEOREM. *Let n be a positive integer whose square-free part is not divisible by a prime of the form $6t+5$. Then there exists a simple loop of order $n+1$ with the cross inverse property. This loop may be made into a neofield by identifying each element except the identity with an appropriate automorphism of the loop.*

The simplicity of the loop mentioned in this theorem is due to a theorem of Bruck, which states, in part, that a finite loop with a transitive automorphism group is either simple or associative.*

There are several variations of our construction which should also be pointed out. First of all, if G is an affine plane with n points on a line where $n \equiv 2 \pmod 3$, or any affine plane with infinitely many points on a line, then we may divide the collection of parallel classes of lines into equivalence classes such that each equivalence class contains exactly three parallel classes, and such that the parallel classes in each equivalence class are given a cyclical order. Then if x, y, z are distinct points of G, we let z be the product of x and y whenever the three lines $\overline{xy}$, $\overline{yz}$, $\overline{xz}$ belong respectively to distinct parallel classes C_1, C_2, C_3 which constitute an equivalence class, and which are arranged here in the correct cyclical order. Developing this situation like the previous one, we would find that an affine plane with n points on a line leads to a cross inverse property loop of order n^2+1. This loop can be made into a neofield if and only if the affine plane is Desarguesian.

A second variation arises by using a sphere instead of a Euclidean plane in our original construction. If x and y are two points on a sphere S which are not antipodal, we define multiplication between them using r and θ just as in the case of the Euclidean plane. To take care of antipodal points, we adjoin a symbol e with the relations $e \circ e = e$, $e \circ x = x \circ e = x'$, and $x \circ x' = e$ for any pair of antipodal points x and x'. This gives an idempotent quasigroup (it will not be Abelian any more) which may be made into a loop as before, and which will again have the cross inverse property for $r = 1$ and $\theta = \frac{1}{3}\pi$. This time, instead of tessellations, one is led to the three regular polyhedra with triangles as faces, giving three cross inverse property loops of order 5, 8, and 14. The first of these loops has appeared in the literature several times as an example of a loop which is not a group, and the second one turns out to be the quaternion group with minor alterations in its multiplication table. The automorphism groups of these three loops are exactly the same as the symmetry groups of the regular polyhedra from which they arise (although in the case of the icosohedron, one must remember to include the "outer automorphism" of Coxeter†).

Finally, our construction may be generalized by replacing the complex plane by an n-dimensional vector space V. Instead of a fixed complex number a, we choose a fixed $n \times n$ matrix A, and define

$$X \circ Y = X + A(Y - X) = (I - A)X + AY$$

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* See [3].

† See page 106 of [4].

for any two vectors X, Y of V. If both A and $(I-A)$ are nonsingular matrices (to satisfy (i)), then the construction of Abelian quasigroups and neofields proceeds as before, except that we cannot use tessellations for $n>2$.

Since the construction of our class of neofields may suggest the construction used by Paige in [8] to some readers, it might be remarked that if the matrix A is chosen to be the negative of the identity matrix (this means $r=1$, $\theta=\pi$ in the case of the plane), we get neofields of the type mentioned by Paige, but that otherwise ours are new. Like Paige's neofields, the class given here cannot be used to define a projective geometry.

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ON CHROMATIC GRAPHS

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1. Introduction. Let N points be given in general position in space (no three in a line, and no four in a plane). All of the segments joining them in pairs are drawn and then colored, some of the segments red and some of the segments blue. What is the smallest possible number of monochromatic triangles, *i.e.*, triangles whose three sides are of the same color?

This problem was first solved by A. W. Goodman [1], and the answer is given in Theorem 1 below. However, his method is a little complicated. We give here, in Section 2, a new and much simpler proof of his result. Although so far the new method has not been helpful in solving the related open problems mentioned by Goodman [1] and by Greenwood and Gleason [2], there is still the hope that eventually the methods used here will be effective in obtaining new information on chromatic graphs.

Finally, an extract from a letter of P. Erdös is included in Section 3 (with his kind permission) in which he settles one of the questions left open by Goodman [1].

2. The minimum number of monochromatic triangles.

THEOREM 1. *Let B and R be the number of solid blue triangles and solid red triangles respectively. Then in any complete coloring of a configuration of N points, using only the colors red and blue, we have*

$$B + R \geqq \begin{cases} \frac{1}{3}u(u-1)(u-2), & \text{if } N = 2u; \\ \frac{2}{3}u(u-1)(4u+1), & \text{if } N = 4u+1; \\ \frac{2}{3}u(u+1)(4u-1), & \text{if } N = 4u+3; \end{cases} \tag{1}$$

where u is a nonnegative integer. Further, this lower bound is sharp for each positive integer N.

Proof. To each colored configuration (chromatic graph) we will attach a weight W determined as the sum of the weights w_P at each point P. At each point P there are $N-1$ line segments emanating, and hence C_2^{N-1} distinct pairs of segments. To each pair of segments at P we assign the weight 2 if the segments are of the same color, and -1 if they are of different colors. Then w_P, the weight at P, is the sum of the weights for all pairs of distinct line segments at P. The weight of a triangle is the sum of the weights of the pairs of segments intersecting at each vertex. Since the points are in general position, each pair of intersecting segments belongs to precisely one triangle; hence the sum of the weights of all the triangles must be W. Clearly each monochromatic triangle has weight 6, and all of the others have weight 0. Hence $B+R$, the total number of monochromatic triangles, is $\frac{1}{6}W$. We must find the minimum value of $\frac{1}{6}W$ for all possible coloring schemes.

Case 1: $N=2u$. At each point w_P will be a minimum when the maximal number of pairs of segments at P are of different colors. This will occur when k segments are of one color and $k-1$ segments are of the other color. When this occurs

$$w_P = 2C_2^u + 2C_2^{u-1} - u(u-1) = (u-1)(u-2).$$

Hence for any coloring scheme $W \geqq 2u(u-1)(u-2)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{1}{3}u(u-1)(u-2). \tag{2}$$

Case 2: $N=4u+1$. In this case w_P will be a minimum when there are $2u$ segments of each color at P. When this occurs

$$w_p = 4C_2^{2u} - 4u^2 = 4u(u-1).$$

Hence for any coloring scheme $W \geqq (4u+1)4u(u-1)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{2}{3}u(u-1)(4u+1). \tag{3}$$

Case 3: $N=4u+3$. It is not possible to have $2u+1$ segments of each color at each point, for then the number of segments of a given color in the whole

configuration would be $\frac{1}{2}(4u+3)(2u+1)$ and this is not an integer. Accordingly, a lower bound is given by the best possible alternative, namely, $2u+1$ segments of each color at $4u+2$ of the points (all but one), and at the remaining point $2u$ segments of one color and $2u+2$ segments of the other color. At each of the first $4u+2$ points we have

$$w_p = 4C_2^{2u+1} - (2u+1)^2 = 4u^2 - 1.$$

At the remaining point

$$w_p = 2C_2^{2u+2} + 2C_2^{2u} - (2u+2)2u = 4u^2 + 2.$$

Hence for any coloring scheme $W \geqq 4u(u+1)(4u-1)$ and therefore

$$R + B = \tfrac{1}{6}W \geqq \tfrac{2}{3}u(u+1)(4u-1). \tag{4}$$

We will now show that the lower bounds found above are sharp by exhibiting coloring schemes whereby equality is attained in (2), (3) and (4).

Equality holds in (2) if and only if at each point there are u segments of one color and $u-1$ segments of the other. Let the points be numbered $P_1, P_2, \cdots, P_{2u}$. Color the segment P_iP_j red if $i+j$ is odd, and blue if $i+j$ is even. This yields equality in (2).

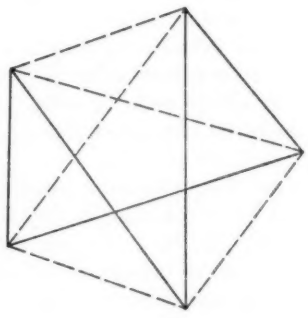

FIG. 1

Equality holds in (3) if and only if at each point there are $2u$ segments of each color, and in (4) if and only if there are $2u+1$ segments of each color at $4u+2$ of the points, and $2u$ segments of one color and $2u+2$ segments of the other at the remaining point. We proceed by induction. Figure 1 shows that a coloring scheme exists which yields equality in (3) for $N=5$. We will assume that a coloring scheme has been found that yields equality in (3) for $N=4u+1$ and show how to obtain one which does the same thing for $N=4u+5$. Furthermore, in doing so we will have found a coloring scheme which yields equality in (4). By assumption, from each point P_i, $i=1, 2, \cdots, 4u+1$, there emanate $2u$ segments of each color. Add the points P_{4u+2} and P_{4u+3} to the configuration. Color the segment $P_{4u+2}P_i$, $i=1, 2, \cdots, 4u+1$ blue if i is odd, red if i is even. Color

the segment $P_{4u+3}P_i$, $i=1, 2, \cdots, 4u+1$ red if i is odd, blue if i is even. Finally color the segment $P_{4u+2}P_{4u+3}$ red. The resulting coloring scheme yields equality in (4) for $N=4u+3$.

Now add the points P_{4u+4} and P_{4u+5} to the configuration. Color the segments $P_{4u+4}P_i$, $i=1, 2, \cdots, 4u+2$ blue if i is odd and red if i is even. Color the segments $P_{4u+5}P_i$, $i=1, 2, \cdots, 4u+2$ red if i is odd and blue if i is even. Finally color the segments $P_{4u+3}P_{4u+4}$, $P_{4u+3}P_{4u+5}$, and $P_{4u+4}P_{4u+5}$ blue, blue, and red respectively. The resulting coloring scheme yields equality in (3) for $N=4u+5$. This completes the proof of Theorem 1.

3. A contribution by P. Erdös. We will call a graph extremal if the equality sign holds in Theorem 1. Goodman has raised the question of whether it is possible to have an extremal graph in which either R or B is zero. In other words, can there be an extremal graph in which all of the monochromatic triangles have the same color? In case N is even the answer is in the affirmative, as was shown in [1], but for odd N the question was left open. We now settle this question by proving that if $N>7$ *is odd, then in any extremal graph both* R *and* B *are positive.* Because of the symmetry it will suffice to prove that $R>0$.

Let us consider first the case that $N=4u+1$ with $u>1$. In an extremal graph, every vertex has $2u$ red lines emanating from it. Let O be one of the vertices, and denote by $P_1, P_2, \cdots, P_{2u}$ the vertices connected with O by a red line. Similarly denote by $Q_1, Q_2, \cdots, Q_{2u}$ the vertices connected with O by a blue line. If $R=0$, no P_i and P_j can be connected with a red line, hence each point P_i is connected to exactly $2u-1$ points Q_j with red lines. Now there are $2u$ red lines at each point Q_j, so u of these red lines must join pairs of points Q_jQ_k. Let us suppose that P_1 is joined by red lines to $Q_1, Q_2, \cdots, Q_{2u-1}$. Then no two of these Q's can be connected with a red line, for otherwise we would have a red triangle. But then Q_{2u} must be joined by u red lines to Q-points, say $Q_1, Q_2, \cdots, Q_u$. But $2u$ red lines emanate from Q_{2u}, so u of these red lines must connect Q_{2u} with points P_i. Let P_2 be one of these points. But P_2 must be connected by red lines with $2u-1$ of the points Q, and, since $u>1$, it must be connected by a red line to one of the points $Q_1, Q_2, \cdots, Q_u$, say Q_i. Then $P_2Q_iQ_{2u}$ is a red triangle.

Assume next that $N=4u+3$. Let O be the exceptional point for the extremal graph, that is at O there are $2u+2$ lines of one color and $2u$ lines of the other color. Hence we must consider two subcases.

Case 1. Suppose that at O there are $2u+2$ red lines and $2u$ blue lines, and at every other point of the graph there are $2u+1$ lines of each color. Let $P_1, P_2, \cdots, P_{2u+2}$ be the points connected to O with red lines and let $Q_1, Q_2, \cdots, Q_{2u}$ be the points connected to O with blue lines. If $R=0$, no P_i and P_j can be connected with a red line. Therefore, since $2u+1$ red lines issue from each P_i, each P_i is connected with each Q_j by a red line. But then each Q_j has $2u+2$ red lines, and this is impossible.

Case 2. Suppose that at O there are $2u$ red lines and $2u+2$ blue lines. Let $P_1, P_2, \cdots, P_{2u}$ be the points connected to O by red lines, and let $Q_1, Q_2, \cdots,$

Q_{2u+2} be the points connected to O by blue lines. If $R=0$, each point P_i is connected by red lines with $2u$ of the points Q_j. At the points Q_j there is a total of $(2u+2)(2u+1)$ red lines, of which none connect with O, and $(2u)^2$ lines connect with points P_i. Hence there are $\frac{1}{2}[(2u+2)(2u+1)-4u^2]=3u+1$ red lines interconnecting points Q_i. Since $N>7$, we have $u>1$. Hence among the $2u+2$ points Q_j there is at least one that is connected to three others by red lines. Suppose these red connections are Q_1Q_2, Q_1Q_3, and Q_1Q_4. If Q_1 is connected by a red line to any P, say P_1, then P_1 cannot be connected with a red line to either Q_2, Q_3, or Q_4, for otherwise we would have a red triangle. But P_1 is connected by red lines to $2u$ of the Q's, and this is impossible. Therefore if $R=0$, Q_1 cannot be connected by a red line to any point P_j. Since $2u+1$ red lines issue from Q_1, it must be connected by red lines with all of the other Q's. So if $R=0$, no other pair Q_i, Q_j $(i>1, j>1)$ can be connected with a red line. But this means there are only $2u+1$ red lines among the Q's, and this contradicts the fact that we have $3u+1$ such lines. Hence $R>0$.

Finally we must consider $N=3$, 5, and 7. The first two cases are trivial since we can have both $R=0$ and $B=0$ (see Fig. 1). In case $N=7$, equation (1) gives $R+B=4$ for the extremal configuration. Now Figure 2 shows that such a chromatic graph is possible with $B=4$ and $R=0$. In that figure the dotted lines denote the blue lines and the solid ones denote the red lines.

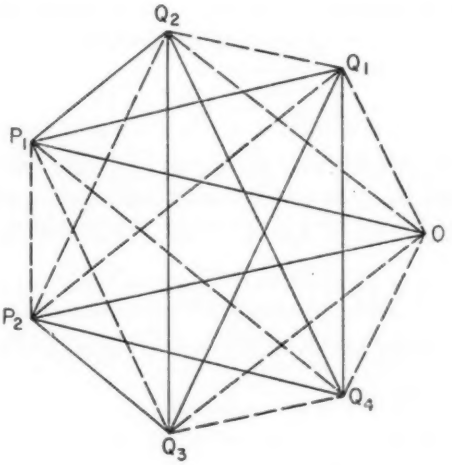

FIG. 2

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THE WEDGE PRODUCT

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1. Introduction. If a suitable product is defined for vectors in two dimensions, the resulting system is isomorphic to the complex numbers, and the usual vector operations can be interpreted in terms of operations on complex numbers. This leads to an elegant alternative approach to topics in which two-dimensional vectors are used. In a similar manner, quaternions may be used in place of three-dimensional vectors. The main disadvantage of complex numbers and quaternions is that they may be used only in two and three dimensions respectively.

The purpose of this note is to discuss another type of algebraic system, the Grassman algebra, which may be used to give a unified approach to several different topics, including vector algebra and its application. Only one new operation, called the wedge product, is needed to express dot products, cross products, etc., in a simple way. The system is easily extended to handle vector analysis. A differential operator is defined in terms of which gradient, curl, etc., may easily be expressed. This operator may also be used, together with the wedge product, to give elegant formulations of other topics such as the Jacobian of a transformation and the integral theorems.

2. The Grassman algebra Λ. Let $B(\epsilon_1, \cdots, \epsilon_n)$, $\epsilon_i = 0, 1$; $i = 1, \cdots, n$, denote 2^n basis vectors of a vector space of 2^n dimensions over the real numbers. Any vector may be written in the form

$$\alpha = \sum_{\epsilon} a(\epsilon_1, \cdots, \epsilon_n) B(\epsilon_1, \cdots, \epsilon_n), \tag{2.1}$$

where the summation is taken over all choices of the ϵ's. The symbols β, γ, etc., will represent similar sums in which b's, c's, etc., replace the a's. The wedge product of two elements $\alpha \wedge \beta = \gamma$ is defined by the identities

$$c(\epsilon_1, \cdots, \epsilon_n) = \sum_{\mu,\nu} \delta \begin{pmatrix} \mu_1, \cdots, \mu_n \\ \nu_1, \cdots, \nu_n \end{pmatrix} a(\mu_1, \cdots, u_n) b(\nu_1, \cdots, \nu_n), \tag{2.2}$$

where the summation is taken over all values of the μ's and ν's, and

$$\begin{aligned} \delta \begin{pmatrix} \mu_1, \cdots, \mu_n \\ \nu_1, \cdots, \nu_n \end{pmatrix} &= 0 \text{ if } \mu_i = \nu_i \text{ for any } i, \\ &= \prod_{i=1}^{n} (-1)^{\nu_i} (\mu_i + \mu_{i+1} + \cdots + \mu_n), \text{ otherwise.} \end{aligned} \tag{2.3}$$

The vector space, together with this wedge product, defines the Grassman algebra Λ [3].

It can be shown that the wedge product is associative, and both distributive laws hold, that is,

(2.4) $(\alpha\wedge\beta)\wedge\gamma=\alpha\wedge(\beta\wedge\gamma)$, $\alpha\wedge(\beta+\gamma)=\alpha\wedge\beta+\alpha\wedge\gamma$, $(\alpha+\beta)\wedge\gamma=\alpha\wedge\gamma+\beta\wedge\gamma$,

so that Λ is a linear associative algebra. The basis elements satisfy the identities

$$(2.5)\quad B(\mu_1, \cdots, \mu_n) \wedge B(\nu_1, \cdots, \nu_n) = \delta\begin{pmatrix}\mu_1, \cdots, \mu_n\\ \nu_1, \cdots, \nu_n\end{pmatrix} B(\mu_1 + \nu_1, \cdots, \mu_n + \nu_n).$$

The wedge product is thus defined by stipulating the rules (2.4) and assuming the identities (2.5) for products of basis elements.

An examination of (2.5) indicates a simpler formulation. Since $B(0, 0, \cdots, 0) \wedge\alpha=\alpha$ for every α, it is convenient to set

$$(2.6)\qquad B(0, 0, \cdots, 0) = 1 = E_0,$$

and identify it with the real number 1. Further set

$$(2.7)\qquad B(0, \cdots, 0, 1, 0, \cdots, 0) = E_i,$$

where the 1 occurs in the ith position. The basis vectors may be expressed in terms of these vectors. In fact,

$$(2.8)\qquad B(\epsilon_1, \cdots, \epsilon_n) = E_{1\epsilon_1} \wedge \cdots \wedge E_{n\epsilon_n}.$$

The terms in which $\epsilon_i=0$ can be omitted.

It follows that $E_0=1, E_1, \cdots, E_n$, form a basis for the linear associative algebra Λ, and that the identities (2.5) may be replaced by the simpler identities.

$$(2.9)\qquad \begin{aligned} E_i \wedge E_j &= -E_j \wedge E_i, && i, j = 1, \cdots, n.\\ E_0 \wedge E_i &= E_i \wedge E_0 = E_i, && i = 1, \cdots, n.\end{aligned}$$

For example, in the case $n=2$, Λ is a 4-dimensional vector space with basis $E_0=1, E_1, E_2, E_1\wedge E_2$, and the wedge product of two elements is given by

$$(a_0 + a_1E_1 + a_2E_2 + a_3(E_1 \wedge E_2)) \wedge (b_0 + b_1E_1 + b_2E_2 + b_3(E_1 \wedge E_2))$$
$$= a_0b_0 + (a_1b_0 + a_0b_1)E_1 + (a_0b_2 + a_2b_0)E_2 + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)(E_1 \wedge E_2).$$

It is convenient to choose the 2^n basis vectors $B(\epsilon_1, \cdots, \epsilon_n)$ of Λ (considered as a vector space) to be unit vectors orthogonal in pairs. Length of vectors will be defined in the usual way (and denoted as usual by $|\alpha|$).

A change of basis for $\mathbf{V}$ (see below) induces a change of basis for Λ in a natural way.

3. An automorphism in Λ. Let $*$ denote the 1-1 mapping which carries α, given by (2.1), into

$$(3.1)\qquad \alpha^* = \sum_{\epsilon} a(\epsilon_1, \cdots, \epsilon_n)B((1 - \epsilon_1, \cdots, 1 - \epsilon_n).$$

It can easily be verified that $*$ is an automorphism of the Grassman algebra.

Let $\mathbf{V}$ denote the n-dimensional subspace spanned by the vectors $E_1, \cdots, E_n$. Every element of $\mathbf{V}$ has the form

$$A = \sum_{i=1}^{n} a_i E_i, \qquad a_i \text{ real.} \tag{3.2}$$

(The letters B, C, etc., will denote vectors in which the a's have been replaced by b's, c's, etc.)

The automorphism $*$ maps the vector E_i on the vector E_i^* given by

$$E_i^* = E_1 \wedge \cdots \wedge E_{i-1} \wedge E_{i+1} \wedge \cdots \wedge E_n. \tag{3.3}$$

These vectors determine a vector space $\mathbf{V}^*$ isomorphic to $\mathbf{V}$ under $*$.

It is clear that $\mathbf{V}$ represents the general n-dimensional real vector space. The algebra Λ in which $\mathbf{V}$ is embedded and the isomorphism $*$ are helpful in studying the properties of $\mathbf{V}$. For example, the scalar product of vectors A and B of $\mathbf{V}$ is given by

$$A \cdot B = (A \wedge B^*)^*. \tag{3.4}$$

4. Geometrical interpretations and applications of the wedge product.

(a) *Volume.* The product $A \wedge B$ of two n-dimensional vectors of $\mathbf{V}$ is a vector in an $\frac{1}{2}n(n-1)$-dimensional orthogonal subspace of Λ. The length of $A \wedge B$, $|A \wedge B|$, represents the area of the parallelogram having edges parallel to A and B.

More generally, let $v(A_1, \cdots, A_r)$ represent the volume of a parallelopiped with edges parallel to $A_1, \cdots, A_r$. The function v is characterised by the properties

(1) $v(A_1, \cdots, A_r) \geqq 0$;
(2) $v(A_1, \cdots, A_{i-1}, A_i + A_j, A_{i+1}, \cdots, A_r) = v(A_1, \cdots, A_r)$ $i \neq j$;
(3) $v(A_1, \cdots, A_{i-1}, aA_i, A_{i+1}, \cdots, A_r) = |a| v(A_1, \cdots, A_r)$, a real;
(4) $v(U_1, \cdots, U_r) = 1$, $U_1, \cdots, U_r$ the edges of an r-dimensional cube; [5].

It is immediate that $|A_1 \wedge \cdots \wedge A_r|$, considered as a function of $A_1, \cdots, A_r$, has these properties and

$$v(A_1, \cdots, A_r) = |A_1 \wedge \cdots \wedge A_r|. \tag{4.1}$$

Let $d(1), \cdots, d\binom{n}{r}$ denote the $r \times r$ determinants that can be formed from the array of components of the vectors $A_1, \cdots, A_r$. Then

$$|A_1 \wedge \cdots \wedge A_r|^2 = d^2(1) + \cdots + d^2\binom{n}{r},$$

so that the volume is easily expressible in terms of the components.

(b) *A generalization of Lagrange's identity.* The volume of the parallelopiped with edges parallel to $A_1, \cdots, A_r$ may be expressed in another simple form. It is easily verified that

$$v(A_1, \cdots, A_r) = \sqrt{(\det (A_i \wedge A_j^*)^*)} \tag{4.2}$$

also satisfies the axioms characterizing volume. Thus, comparing with (4.1),

$$\det(A_i \wedge A_j^*)^* = |A_1 \wedge \cdots \wedge A_r|^2. \tag{4.3}$$

Formula 4.3 is a generalization of the well-known Lagrange identity [2]. In fact, setting $n=3$ and $r=2$ in (4.3) yields

$$\begin{aligned}(a_{11}^2 + a_{12}^2 + a_{13}^2)(a_{21}^2 + a_{22}^2 + a_{32}^2) - (a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23})^2 \\ = (a_{12}a_{23} - a_{13}a_{22})^2 + (a_{13}a_{21} - a_{11}a_{13})^2 + (a_{11}a_{22} - a_{12}a_{21})^2,\end{aligned} \tag{4.4}$$

where

$$A_i = (a_{i1}, a_{i2}, a_{i3}), \qquad i = 1, 2.$$

(c) *Linear independence.* The vectors $A_1, \cdots, A_r$ are linearly independent if and only if the corresponding parallelepiped has nonzero volume. Thus: *The vectors $A_1, \cdots, A_r$ are linearly independent if and only if $A_1 \wedge \cdots \wedge A_r \neq 0$.*

This theorem may be used to deduce the equation of an r-space in compact form. Let P, P_0, P_1, etc., denote position vectors of points of E_n, Euclidean space of n dimensions. If the points $P_0, P_1, \cdots, P_r$ determine an r-space S, the vectors $P-P_0, P-P_1, \cdots, P-P_r$ will determine a parallelepiped of zero volume if and only if P lies in S. Thus the equation of S is

$$(P - P_0) \wedge (P - P_1) \wedge \cdots \wedge (P - P_r) = 0. \tag{4.5}$$

(d) *Vector algebra* [2]. The scalar product $A \cdot B$ of elements of $\mathbf{V}$ may be expressed in terms of the wedge product:

$$A \cdot B = (A \wedge B^*)^*. \tag{4.6}$$

In line with the discussion in (c), the hyperplane through P_1 having normal A is given by $(P-P_1) \wedge A^* = 0$.

In case $\mathbf{V}$ is 3-dimensional, vector products are usually defined. In terms of the wedge product

$$A \times B = (A \wedge B)^*. \tag{4.7}$$

Thus: *Nonzero vectors A, B are perpendicular (parallel) if and only if $A \wedge B^* = 0$ $(A \wedge B = 0)$.*

The other vector products and identities can be translated into forms involving the wedge product. For example

$$A \times B \cdot C = (A \wedge B \wedge C)^*, \qquad (A \times B) \times C = ((A \wedge B)^* \wedge C)^*;$$

and the vector identity $(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$ is equivalent to the wedge identity

$$(A \wedge B)^* \wedge C = (C^* \wedge B)^* A^* + (A \wedge C^*)^* B^*.$$

(e) *Determinants* [5]. Let $A_i = a_{i1}E_1 + \cdots + a_{in}E_n$ $(i = 1, \cdots, n)$ denote

the vectors corresponding to the rows of the $n\times n$ determinant $d=|a_{ij}|$. Then $d=(A_1\wedge\cdots\wedge A_n)^*$. The theory of determinants is easily developed from this point of view. For example, to deduce the usual expansions in terms of cofactors, set $\overline{A}_i=\bar{a}_{i1}E_1+\cdots+\bar{a}_{in}E_n$, where $\overline{A}_i^*=(-1)^{i+1}A_1\wedge\cdots\wedge A_{i-1}\wedge A_{i+1}\wedge\cdots\wedge A_n$. The numbers $\bar{a}_{ij}$ are the cofactors of a_{ij} $(i,j=1,\cdots,n)$, and

$$\begin{aligned} d^* &= A_1\wedge\overline{A}_1=\cdots=A_n\wedge\overline{A}_n \\ &= a_{i1}\bar{a}_{i1}+a_{i2}\bar{a}_{i2}+\cdots+a_{in}\bar{a}_{in} \qquad (i=1,\cdots,n). \end{aligned}$$

The general Laplace expansion may be deduced in the same way by considering the products $A_{i1}\wedge\cdots\wedge A_{ir}$.

The vector properties lead to the usual rules for determinants and methods of simplification. As an example, consider the Vandermonde determinant

$$\Delta=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Let $A=E_1+aE_2+a^2E_3$, $B=E_1+bE_2+b^2E_3$, $C=E_1+cE_2+c^2E_3$,

$$B_1=\frac{B-A}{b-a}=E_2+(b+a)E_3,\ C_1=\frac{C-A}{c-a}=E_2+(c+a)E_3,\ C_2=\frac{C_1-B_1}{c-b}E_3.$$

Then

$$\begin{aligned} \Delta^*&=A\wedge B\wedge C=A\wedge(b-a)B_1\wedge(c-a)C_1=A\wedge(b-a)B_1\wedge(c-a)(c-b)C_2 \\ &=(b-a)(c-a)(c-b)E_1\wedge E_2\wedge E_3. \end{aligned}$$

5. Augmented vector fields. With every point $(x_1,\cdots,x_n)$ of E_n associate a vector $A(x_1,\cdots,x_n)$ of **V**. This defines a *vector field* A in E_n. Continuity, partial derivatives, etc., are easily defined. More generally, an element $\alpha(x_1,\cdots,x_n)$ of Λ can be associated with every point of E_n. This defines an *augmented vector field*. Derivatives are defined in the natural way [2]. Let D denote the set of augmented vector fields which have continuous partial derivatives in some region.

A transformation ∇ may be defined in D, which generalizes the gradient operator. Let ∇f denote the gradient of f when f denotes a real-valued function of n variables, that is,

$$\nabla f=\frac{\partial f}{\partial x_1}E_1+\cdots+\frac{\partial f}{\partial x_n}E_n. \tag{5.1}$$

Notice that $\nabla x_i=E_i$, $i=1,\cdots,n$. Let

$$\nabla E_i=0, \qquad i=1,\cdots,n, \tag{5.2}$$

and assume

$$(5.3)\qquad \nabla(\alpha+\beta)=\nabla\alpha+\nabla\beta,\qquad \nabla(\alpha\wedge\beta)=\nabla\alpha\wedge\beta+\alpha\wedge\nabla\beta.$$

In particular, if f is defined as above, $f\wedge\alpha=f\alpha$ so that

$$(5.4)\qquad \nabla(f\alpha)=\nabla f\wedge\alpha+f\nabla\alpha.$$

The relations (5.1), (5.2), (5.3), and (5.4) define $\nabla\phi$ for every element ϕ of D. We now give some applications of ∇.

6. Applications of augmented vector fields.

(a) *Jacobian of a transformation* [1]. Consider the transformation of coordinates defined by the equations

$$(6.1)\qquad x_i=f_i(u_1,\cdots,u_n),\qquad i=1,\cdots,n.$$

If the functions f_i, $i=1,\cdots,n$, have continuous derivatives, and if the Jacobian J is not zero, the variables u_i, $i=1,\cdots,n$, are continuous differentiable functions of $x_1,\cdots,x_n$, and u_i, $i=1,\cdots,n$, are well defined. Applying ∇ to the equations (6.1) yields $E_1,\cdots,E_n$ in terms of $\nabla u_1,\cdots,\nabla u_n$. A straightforward calculation yields

$$(6.2)\qquad \nabla x_1\wedge\cdots\wedge\nabla x_n=E_1\wedge\cdots\wedge E_n=J(\nabla u_1\wedge\cdots\wedge\nabla u_n).$$

Equation (6.2) may be used to evaluate J. To illustrate the calculations involved, consider the transformation to spherical polar coordinates ρ, θ, ϕ in E_3, given by

$$(6.3.)\qquad x=\rho\cos\theta\cos\phi,\qquad y=\rho\cos\theta\sin\phi,\qquad z=\rho\sin\theta.$$

Let $\rho_1=\rho\cos\theta$, so that $x=\rho_1\cos\phi$, $y=\rho_1\sin\phi$, and $\nabla x\wedge\nabla y=(\nabla\rho_1-\rho_1\sin\phi\nabla\phi)\wedge(\nabla\rho_1\sin\phi+\rho_1\cos\phi\nabla\phi)=\rho_1\nabla\rho_1\wedge\nabla\phi$. Similarly, $\nabla\rho_1\wedge\nabla z=\rho\nabla\rho\wedge\nabla\theta$. Thus, $\nabla x\wedge\nabla y\wedge\nabla z=(\rho_1\nabla\rho_1\wedge\nabla\phi)\wedge\nabla z=-\rho_1(\nabla\rho_1\wedge\nabla z)\wedge\nabla\phi=-\rho_1\rho\nabla\rho\wedge\nabla\theta\wedge\nabla\phi$. The Jacobian is thus $-\rho_1\rho=-\rho^2\cos\theta$.

To illustrate the type of calculations involved in the generalization to n dimensions, consider the transformation to spherical polar coordinates in n dimensions. This transformation, which is of importance in statistics, is given by

$$(6.4)\qquad x_i=\rho\cos\theta_1\cdots\cos\theta_{n-i}\sin\theta_{n-i+1},\quad \theta_0=0,\ \theta_n=\tfrac{1}{2}\pi;\qquad i=1,\cdots,n.$$

Let $\rho_i=\rho\cos\theta_1\cdots\cos\theta_i$, $i=1,\cdots,n-2$, and set $\rho_0=\rho$. Then $x_1=\rho_{n-2}\cos\theta_{n-1}$, $x_i=\rho_{n-i}\sin\theta_{n-i+1}$, and $\rho_{n-i+1}=\rho_{n-i}\cos\theta_{n-i+1}$, $i=2,\cdots,n$. As before,

$$(6.5)\qquad \nabla x_1\wedge\nabla x_2=\rho_{n-2}\nabla\rho_{n-2}\wedge\nabla\theta_{n-1},$$

$$(6.6)\qquad \nabla x_i\wedge\nabla\rho_{n-i+1}=\rho_{n-i}\nabla\rho_{n-i}\wedge\nabla\theta_{n-i+1},\qquad i=2,\cdots,n.$$

Using (6.5) and (6.6) in an induction on r, it easily follows that

$$(6.7)\qquad \begin{aligned}&\nabla x_1\wedge\cdots\wedge\nabla x_r\\ &\quad=(-1)^{(r-1)(r-2)/2}\rho_{n-2}\cdots\rho_{n-r}\nabla\rho_{n-r}\wedge\nabla\theta_{n-(r-1)}\wedge\cdots\wedge\nabla\theta_{n-1}.\end{aligned}$$

In particular, taking $r=n$, the Jacobian of the transformation is given by

$$J = (-1)^{(r-1)(r-2)/2}\rho_{n-2}\rho_{n-3}\cdots\rho_0$$
$$= (-1)^{(r-1)(r-2)/2}\rho^{n-1}\cos^{n-2}\theta_1\cos^{n-3}\theta_2\cdots\cos\theta_{n-2}.$$

This Jacobian is used in developing the probability density function of χ^2 [4].

(b) *Vector analysis.* Applying ∇ to the elements of D corresponding to $\mathbf{V}$ and $\mathbf{V}^*$ of Λ in three dimensions is equivalent to the ordinary operations in vector analysis [2]. In fact,

$$(6.8)\quad \nabla A = (\text{curl } A)^*,\quad \nabla A^* = (\text{div } A)^*,\quad \nabla(\nabla f) = 0,\quad \nabla(\nabla f)^* = (\text{Lap } f)^*$$

Any calculations involving curl, div, etc., can thus be carried out in terms of ∇ in any coordinate system.

For example, consider the transformation to spherical polar coordinates given by (6.3). Applying ∇ to the equations yields equations for $\nabla\rho$, $\nabla\theta$, $\nabla\phi$. It is easily verified that

$$\begin{aligned}\nabla\rho &= \cos\theta\cos\phi E_1 + \cos\theta\sin\phi E_2 + \sin\theta E_3,\\ \rho\nabla\theta &= -\sin\theta\cos\phi E_1 - \sin\theta\sin\phi E_2 + \cos\theta E_3,\\ \rho\cos\theta\nabla\phi &= -\sin\phi E_1 + \cos\phi E_2;\end{aligned}\tag{6.9}$$

$$\begin{aligned}&\nabla\rho\wedge(\nabla\rho)^* = E_1\wedge E_2\wedge E_3,\quad \nabla\rho\wedge(\nabla\theta)^* = \nabla\rho\wedge(\nabla\phi)^* = 0,\\ &\nabla\theta\wedge(\nabla\theta)^* = \rho^{-2}(E_1\wedge E_2\wedge E_3),\quad \nabla\theta\wedge(\nabla\phi)^* = 0,\\ &\nabla\phi\wedge(\nabla\phi)^* = \rho^{-2}\sec^2\theta\,(E_1\wedge E_2\wedge E_3).\end{aligned}\tag{6.10}$$

The Laplacians of ρ, θ, and ϕ can now be evaluated. For example,

$$\nabla(\nabla\rho)^* = \nabla(\cos\theta\cos\phi)(E_2\wedge E_3) + \nabla(\cos\theta\sin\phi)(E_3\wedge E_1)$$
$$+ \nabla\sin\theta(E_1\wedge E_2) = (2/\rho)(E_1\wedge E_2\wedge E_3),$$

so that $\text{Lap}\,\rho = 2/\rho$ by (6.8). In a similar manner, Lap θ and Lap ϕ may be calculated.

$$(6.11)\qquad \text{Lap}\,\rho = 2/\rho,\qquad \text{Lap}\,\theta = -(\tan\theta)/\rho^2,\qquad \text{Lap}\,\phi = 0.$$

Now, consider any function V of ρ, θ, ϕ which is twice differentiable. Using subscripts to indicate partial derivatives

$$(\nabla V)^* = V_\rho(\nabla\rho)^* + V_\theta(\nabla\theta)^* + V_\phi(\nabla\phi)^*;$$

so that, using (6.10)

$$\begin{aligned}\nabla(\nabla V)^* &= \nabla V_\rho\wedge(\nabla\rho)^* + \nabla V_\theta\wedge(\nabla\theta)^* + (\nabla V_\phi)\wedge(\nabla\phi)^*\\ &\quad + V_\rho\nabla(\nabla\rho)^* + V_\theta\nabla(\nabla\theta)^* + V_\phi\nabla(\nabla\phi)^*\\ &= V_{\rho\rho}\nabla\rho\wedge(\nabla\rho)^* + V_{\theta\theta}\nabla\theta\wedge(\nabla\theta)^* + V_{\phi\phi}\nabla\phi\wedge(\nabla\phi)^*\\ &\quad + V_\rho\nabla(\nabla\rho)^* + V_\theta\nabla(\nabla\theta)^* + V_\phi\nabla(\nabla\phi)^*.\end{aligned}$$

Finally, using (6.8), (6.10), (6.11) yields

$$\text{(6.12)} \qquad \text{Lap } V = V_{\rho\rho} + \frac{1}{\rho^2} V_{\theta\theta} + \frac{\sec^2\theta}{\rho^2} V_{\phi\phi} + \frac{2}{\rho} V_\rho - \frac{\tan\theta}{\rho^2} V_\theta.$$

(c) *Substitution in triple integrals* [1]. If the transformation (6.1) is applied to the n-tuple integral of $F(x_1, \cdots, x_n)$ over the region R, it is well known that, under suitable assumptions,

$$\int_R \cdots \int F(x_1, \cdots, x_n) dx_1 \cdots dx_n = \int_{\bar{R}} \cdots \int \bar{F}(u_1, \cdots, u_n) J du_1 \cdots du_n,$$

where

$$\text{(6.13)} \qquad \bar{F}(u_1, \cdots, u_n) = F[f_1(u_1, \cdots, u_n), \cdots, f_n(u_1, \cdots, u_n)],$$

and $\bar{R}$ is the region corresponding to R under the transformation. A comparison with (6.2) suggests the use of $\nabla x_1 \wedge \cdots \wedge \nabla x_n$ instead of $dx_1 \cdots dx_n$ in the symbol for the integral. Changes in coordinates could then be made by expressing each of the symbols in the new coordinates, as in the case of ordinary Riemann integrals, that is,

$$\text{(6.14)} \qquad \begin{aligned} &\int_R \cdots \int F(x_1, \cdots, x_n) \nabla x_1 \wedge \cdots \wedge \nabla x_n \\ &\qquad = \int_{\bar{R}} \cdots \int \bar{F}(u_1, \cdots, u_n) J \nabla u_1 \wedge \cdots \wedge \Delta u_n, \end{aligned}$$

using (6.2) and (6.13).

(d) *Integral theorems* [2]. The previous remarks indicate the advantage of using ∇x, ∇y, ∇z instead of dx, dy, and dz in integration in E_3. Define line and surface integrals of other elements of D by

$$\int_C A = \int_C a_1 dx + a_2 dy + a_3 dz, \qquad \iint A^* = \int_S a_1 dy dz + a_2 dz dx + a_3 dx dy.$$

For a suitably well-behaved region R whose boundary is a surface S, the divergence theorem states that $\iint_S A^* = \iiint_R \nabla A^*$. Similarly, for a suitably well-behaved orientable surface R bounded by a curve C, Stokes' theorem states that $\int_C A = \iint_R \nabla A$.

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THE COMPLEX SUM OF DIVISORS

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0. Introduction. The gaussian integers have been the subject of numerous investigations. In fact, for every property of the natural numbers one can ask if the same is true for the gaussian integers, or for other unique factorization domains. One of the first examples of this is Dirichlet's extension of his theorem on primes in arithmetic progressions to the complex case. Many other topics have been considered, such as quadratic reciprocity, quadratic forms, the Pell equation, sums of squares, the Fermat problem and asymptotic laws.

Natural questions certainly arise about number theory functions such as the totient function and the number-of-divisors function; and these have been defined for the gaussian integers and shown to have analogous properties to their real counterparts. These last functions are "counting" functions, from the complex numbers to the reals. Gegenbauer [1] has defined a function in which the norms of the divisors are summed. One can ask if there is a complex sum-of-divisors function, from the complexes to the complexes. This paper takes up the definition and simplest properties of such a function. Also treated are complex Mersenne and complex perfect numbers. The methods are elementary.

The method used to define the complex σ-function is to first define it for powers of primes and then to extend the definition multiplicatively. For powers of primes we try the guess $\sigma(\pi^n)=1+\pi^*+\cdots+\pi^{*n}$, where π^* is an associate of π, and ask: How does the would-be definition of $\sigma(\pi^n)$ depend on the choice of the associate π^*. The remarkable answer to this question is that for the reasonable property $|\sigma(\pi^n)|\geqq|\pi^n|$, the associate chosen must lie in the right half-plane. An opposite inequality results from choices in the left half-plane.

1. Definition of $\sigma(a+bi)$. There are, *a priori*, infinitely many possible definitions of the function $\sigma(a+bi)$. For one can choose for each divisor of $a+bi$ any one of the four associates and add them up. For instance, one could define $\sigma(1+i)$ as $\alpha+\beta$, where $\alpha\in\{1,-1,i,-i\}$ and $\beta\in\{1+i,1-i,-1+i,-1-i\}$, giving 16 choices for the definition, and similarly for other gaussian integers. From among this chaos of possibilities it is possible to select a well-behaved function, having the natural property that $|\sigma(a+bi)|\geqq|a+bi|$. In the following, I will use Greek letters for gaussian integers, π denoting a gaussian prime.

We wish, first of all, to have a multiplicative function, *i.e.*, $\sigma(\alpha\cdot\beta)=\sigma(\alpha)\cdot\sigma(\beta)$ for $(\alpha,\beta)=1$. To achieve this, we set $\sigma(\epsilon)=1$, for ϵ a unit. Then we set $\sigma(\pi^k)=\sigma((i\pi)^k)=\sigma((-\pi)^k)=\sigma((-i\pi)^k)=$ a certain number defined below, independent of the choice of associate of π. Then if $\eta=\prod\pi_i^{k_i}$, we set $\sigma(\eta)=\prod\sigma(\pi_i^{k_i})$. By these means, we clearly obtain a multiplicative sum-of-divisors function.

It now remains to define $\sigma(\pi^k)$. This definition is based on the following lemma.

LEMMA. *For complex integers α, such that $|\alpha|>1$,*

$$|1+\alpha+\cdots+\alpha^n| \geqq |\alpha^n|, \text{ for } \Re(\alpha)>0; \tag{1}$$

$$|1+\alpha+\cdots+\alpha^n| \leqq |\alpha^n|, \text{ for } \Re(\alpha)<0. \tag{2}$$

Proof. Let $\alpha=a+bi$, then for $\Re(\alpha)>0$, setting the norm of $\eta=\|\eta\|$,

$$\left\|\frac{\alpha^{n+1}-1}{\alpha-1}\right\| = \left\|\frac{\alpha^{n+1}-1}{a+bi-1}\right\| > \left\|\frac{\alpha^{n+1}-1}{a+bi}\right\| = \left\|\frac{\alpha^{n+1}-1}{\alpha}\right\| \geqq \|\alpha^n\| - 1/\|\alpha\|.$$

Thus $\|1+\alpha+\cdots+\alpha^n\|>\|\alpha^n\|-1/\|\alpha\|$; but since the left-hand side is an integer and $\|\alpha^n\|$ is an integer, we can neglect the term $1/\|\alpha\|$ (being <1) if we replace $>$ by $\geqq$. Thus (1) is proved for norms, and hence is true for absolute values. The proof of (2) is similar.

This lemma was discovered geometrically by plotting values of $1+\alpha+\cdots+\alpha^n$ and values of α^n on the same sheet of paper for α's in various parts of the plane.

According to the lemma, if we fix on an associate π^* of π with $\Re(\pi^*)>0$, and set $\sigma(\pi^k)=1+\pi^*+\cdots+\pi^{*k}$, we will have $|\sigma(\pi^k)| \geqq |\pi^k|$. To fix our ideas, the first quadrant is thought of as containing the positive real semiaxis, but not the imaginary semiaxis. Also, from the two associates lying in the right half plane, we arbitrarily select the π^* lying in the first quadrant. Other possibilities are discussed later in Section 4. The foregoing paragraphs are summed up in the following definition.

DEFINITION. *Let η be a gaussian integer. To calculate $\sigma(\eta)$, factor η into a product of powers of distinct primes, $\eta=\epsilon\prod\pi_i^{k_i}$, ϵ a unit, each π_i lying in the first quadrant, then $\sigma(\eta)=\prod(\pi_i^{k_i+1}-1)/(\pi_i-1)$.*

Clearly σ is multiplicative and $|\sigma(\eta)| \geqq |\eta|$. Where the domains of the rational powers of primes and gaussian powers of primes overlap, the sum-of-divisors functions coincide. Thus in a natural sense, the complex σ-function is an extension of the real σ-function. However, for instance, for the real σ-function, $\sigma(2)=3$, but for the complex σ-function, $\sigma(2)=2+3i$. A small table of the complex σ-function is given in Table 1 below.

For the natural numbers we have the well-known laws of parity:

$$\text{odd} + \text{odd} = \text{even} + \text{even} = \text{even},$$

$$\text{odd} + \text{even} = \text{even} + \text{odd} = \text{odd}.$$

It is an unusual fact that these laws of parity hold also for the gaussian integers, where one defines: α is even iff $1+i$ divides α. The proof of the parity laws consists in checking the 16 cases arising from the fact that $a+bi$ is odd iff a and b have different real parity.

Using these laws of parity one can prove, in a manner similar to the real case, the following theorem: $\sigma(\alpha)$ is odd iff α is a square or $1+i$ times a square or an associate of such.

TABLE 1

$\sigma(a+bi)$

a \ b	0	1	2	3	4	5	6
1	1	$2+i$	$2+2i$	$5+5i$	$2+4i$	$6+8i$	$2+6i$
2	$2+3i$	$3+i$	$5i$	$3+3i$	$-2+10i$	$3+5i$	$-5+15i$
3	4	$2+6i$	$4+2i$	$8+4i$	$6+5i$	$9+7i$	$8+8i$
4	$-4+5i$	$5+i$	$3+11i$	$-1+6i$	$-8+i$	$5+5i$	$-3+15i$
5	$4+8i$	$3+9i$	$6+2i$	$10i$	$6+4i$	$20i$	$6+6i$
6	$8+12i$	$7+i$	$-10+10i$	$12+4i$	$2+16i$	$7+5i$	$20i$

2. Complex Mersenne numbers. The class of numbers $(q^p-1)/(q-1)$, with p and q rational primes, I call general Mersenne numbers. If $q=2$, these are called Mersenne numbers. Similarly, the numbers $(\pi^p-1)/(\pi-1)$, π a gaussian prime and p a rational prime, I call the general complex Mersenne numbers, and if $\pi=1+i$, these are called complex Mersenne numbers, and are denoted as M_p. These classes of numbers are useful in the study of σ-functions.

Table 2 gives the real and imaginary parts of M_p, where $h=\frac{1}{2}(p-1)$.

TABLE 2

	$p\equiv 1(8)$	$p\equiv 3(8)$	$p\equiv 5(8)$	$p\equiv 7(8)$
$\Re(M_p)$	2^h	2^h	-2^h	-2^h
$\Im(M_p)$	$-(2^h-1)$	2^h+1	2^h+1	$-(2^h-1)$

If $p\equiv\pm 1 \pmod 8$, then $\|M_p\|=2^{2h+1}-2^{h+1}+1$, and if $p\equiv\pm 3 \pmod 8$, then $\|M_p\|=2^{2h+1}+2^{h+1}+1$. On the other hand,

$$2^{2p}+1=(2^{2h+1}-2^{h+1}+1)(2^{2h+1}+2^{h+1}+1),$$

so that $\|M_p\|$ divides $2^{2p}+1$ in all cases (p an odd prime). The numbers $2^{2p}+1$ have been studied by Cunningham [2], and others. Mr. John Brillhart is at present searching for factors of these numbers, and will present a complete report shortly.

3. Complex perfect numbers. One possible definition of complex perfect numbers would be to say that α is perfect iff $\sigma(\alpha)=2\alpha$. These numbers, however, can be regarded as multiply perfect. As $1+i$ is the "prime of least norm," we take the definition: α is perfect iff $\sigma(\alpha)=(1+i)\alpha$. We have, using this definition, the following analog to Euclid's theorem on perfect numbers:

THEOREM. *If M_p is a complex Mersenne prime and if $p \equiv 1 \pmod 8$, then $(1+i)^{p-1} \cdot M_p$ is perfect.*

Proof. $M_p = -i[(1+i)^p - 1]$ and

$$\sigma[(1+i)^{p-1} \cdot M_p] = -i[(1+i)^p - 1]\sigma[(1+i)^p - 1].$$

To find $\sigma[(1+i)^p - 1]$, we pick the associate π, with $\Re(\pi) > 0$ and $\Im(\pi) > 0$. It is easy to see that this associate is $[(1+i)^p - 1]$ itself iff $p \equiv 1 \pmod 8$. In this case $\sigma[(1+i)^p - 1] = (1+i)^p$, and the theorem is proved.

For $p \equiv 1 \pmod 8$, the M_p mentioned in the previous section are all composite for $p < 250$ except for the cases $p = 73, 241$, which have recently been shown to be prime by D. H. Lehmer and John Brillhart.

In the real case, we have also the converse to the analog of the above theorem. The proofs of this converse do not appear to generalize to the complex case. However, we do have the following theorem.

THEOREM. *If $a+bi$ is odd then $(1+i)^{k-1} \cdot (a+bi)$ can be perfect only if $k \equiv 0$ or $k \equiv \pm 1 \pmod 8$.*

Proof. Table 3 gives the real and imaginary parts of $(1+i)^k$, where $h = [\frac{1}{2}k]$.

TABLE 3

$k \equiv$	0	1	2	3	4	5	6	7	(mod 8)
$\Re$	2^h	2^h	0	-2^h	-2^h	-2^h	0	2^h	
$\Im$	0	2^h	2^h	2^h	0	-2^h	-2^h	-2^h	

If $(1+i)^{k-1} \cdot (a+bi)$ is perfect, then we have $\sigma[(1+i)^{k-1} \cdot (a+bi)] = -i[(1+i)^k - 1]\sigma(a+bi) = (1+i)^k \cdot (a+bi)$. However, $|(1+i)^k - 1| > |(1+i)^k|$ iff $k \equiv 2, 3, 4, 5, 6 \pmod 8$, and always $|\sigma(a+bi)| \geqq |a+bi|$; hence $|-i[(1+i)^k - 1]\sigma(a+bi)| > |(1+i)^k \cdot (a+bi)|$ and the theorem is proved.

The complex perfect numbers are included in the class of numbers for which $\|\sigma(\alpha)\| = 2\|\alpha\|$. These numbers I call norm-perfect. A curiosity of this class is the number $2+i$, an odd norm-perfect number.

Abundant and deficient numbers can be defined as numbers α for which $\|\sigma(\alpha)\| > 2\|\alpha\|$, and $\|\sigma(\alpha)\| < 2\|\alpha\|$, respectively. Thus $1+i$ is abundant, $1+2i$ is deficient. One could also, in an obvious manner, define multiply-perfect and norm-multiply-perfect numbers.

4. Other possible definitions. In the definition of Section 1, we chose, of the two associates of π with positive real part, the one lying in the first quadrant. One could define other σ-functions by taking associates in the fourth quadrant, or even mixing the choice of associates in the first and fourth quadrants. This would certainly be justified, if by so doing one could prove, say, an analog of

the Euler pentagonal-number recurrence formula. So far, I have found no such relation. The choice made in this paper was made on the basis of simplicity. It makes no difference in the theory of complex Mersenne numbers, since, if we change the choice, we would be dealing merely with the conjugate numbers.

However, this could make a difference in the theory of perfect numbers. For instance, if one defined: α is perfect iff $\sigma(\alpha)=(1-i)\alpha$, and let $\sigma((1+i)^k)=-i[(1-i)^{k+1}-1]$, then $\sigma[(1-i)^6(-i)(7+8i)]=(1-i)^7(-i)(7+8i)$ and we would have a perfect number corresponding to an associate of M_7. One could possibly force a number to be perfect by mixing the choice of associates in the right and left half-planes, but this has little significance.

Another natural possibility for defining a sum-of-divisors function would be to use the second part of the lemma in Section 1, and construct a σ-function satisfying $|\sigma(\alpha)| \leqq |\alpha|$. One can even define in the real case a function of this type, setting $\sigma^*(p^k)=1-p+p^2-\cdots+(-1)^k p^k$.

5. Conclusions. It has been shown in this paper how to define a sum-of-divisors function for the gaussian integers. An important property of this function is that $|\sigma(\alpha)| \geqq |\alpha|$. Much of the real σ-function theory, such as Mersenne and perfect numbers, can be carried over to the complex case.

The strictness of the above inequality was checked for powers of primes up to the ninth power. A proof of this strictness for all powers would certainly be desirable. One would also like to know if all even complex perfect numbers are of "Euclid's" type, and if there are any odd complex perfect numbers.

I wish to thank Dr. D. H. Lehmer for many thoughtful suggestions during the preparation of this paper.

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DESIGN OF MIXED DOUBLES TOURNAMENTS

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1. Introduction. This paper will discuss combinatorial problems which arise in designing mixed doubles tournaments. In each match of a mixed doubles tournament two teams, each containing one man and one woman, compete with one another (as in tennis, bridge, etc.). In the literature on the design of doubles tournaments (see F. Scheid [5] and G. W. Beynon [1]) the added requirement that each team be a man-woman pair seems not to have been considered.

For scoring purposes, ordinary doubles tournaments are designed to have a high degree of symmetry; *e.g.*, each player appears once as partner to every other player and twice as opponent to every other player. In a mixed tournament only much weaker symmetries are possible. In the designs which follow each player is to meet many other players without undue repetitions. One constraint will be

I. *No two players shall appear together in more than one match of the tournament (regardless of whether they play on the same or on opposite teams in that match).*

If I limits the number of matches in the tournament too severely, an alternative constraint is

II. *No two players of the same sex shall appear together in more than one match; two players of opposite sex can play in* 0, 1, *or* 2 *matches but in the last case they must appear once as teammates and once as opponents.*

In designing a tournament subject to I or II we will try to maximize the number T of matches in the tournament. It will be assumed that there is the same number n of men as of women in the tournament. If I is adopted, then

$$T \leqq [\tfrac{1}{2}n[\tfrac{1}{2}n]]. \tag{1}$$

To prove (1), note that each man can appear in at most $[\frac{1}{2}n]$ matches and that

$$2T = \sum_{\text{man } i} \{\text{number of matches containing man } i\}.$$

Similarly, if II is adopted, then

$$T \leqq \tfrac{1}{2}n(n-1) \tag{2}$$

for now each man can appear in at most $n-1$ matches.

The bounds (1), (2) are generally found to be very close to the maximum values of T and are achieved for infinitely many values of n. Whenever (1) is attained for even n, each player meets $\frac{1}{2}n$ players of the same sex and all n players of opposite sex. Whenever (2) is attained, each player meets all other players of the same sex and at least $n-1$ players of opposite sex.

2. Design for I with $n=12k\pm2$. The design of this section is based on a suggestion of Conyers Herring. Let $N=\frac{1}{2}n=6k\pm1$. Number the $2N$ men using the labels $1, \cdots, N, 1^*, \cdots, N^*$. Also number the women with the same set of labels. Each match of the tournament will be described by a quadruple (m_1, m_2, w_1, w_2) where m_1, m_2, w_1, w_2 are the labels given to the man on team #1, the man on team #2, $\cdots$, the woman on team #2.

Pick three integers a, b, c such that each of $a, b, c, a-b, b-c, c-a$ is relatively prime to N. For example, with $N=6k\pm1$, $a=1$, $b=2$, $c=3$ suffice. The tournament is a two-parameter family of matches $M(i, j)$, $i=1, \cdots, N$,

$j=1, \cdots, N$. Match $M(i, j)$ is the quadruple

$$M(i, j) = (i, (i + aj)^*, i + bj, (i + cj)^*),$$

where the additions are modulo N. Note, in this design, that two players of the same sex appear together only if one has a starred number and the other has an unstarred number.

To verify that no pair of players meets twice (Condition I) one may consider pairs of players of six possible kinds: a pair of men (m and m^*), a pair of women (w and w^*), and four kinds of man-woman pairs (m and w, m and w^*, m^* and w, m^* and w^*). Six necessary and sufficient conditions, each to prevent one of the six pairs from meeting twice, are that a, $c-a$, b, c, $b-a$, and $c-a$ be relatively prime to N. Proofs in the six cases are similar. For example, if (contrary to I) the pair m^*, w appears both in $M(i, j)$ and in $M(I, J)$, then

$$m^* \equiv i + aj \equiv I + aJ \pmod{N}, \qquad w \equiv i + bj \equiv I + bJ \pmod{N};$$

whence $(a-b)(j-J) \equiv 0 \pmod{N}$. Since $a-b$ is relatively prime to N, $j \equiv J \pmod{N}$ and then $i \equiv I \pmod{N}$, a contradiction. Conversely, if D divides both N and $a-b$, then the pair m^*, w which appears in $M(i, j)$ reappears in $M(i-a(N/D), j+(N/D))$.

Of the six numbers, a, b, c, $a-b$, $b-c$, $c-a$, at least one is even and at least one is divisible by 3. Then the only N's which can be relatively prime to all six numbers are those of the form $6k \pm 1$. Thus this design does not generalize to other values of N.

In this design $[\frac{1}{2}n[\frac{1}{2}n]] = N^2$, the number of matches; thus the bound (1) is attained. Moreover, as mentioned in the introduction, each player will meet all players of the opposite sex since n is even.

In some tournaments it is required that the matches be arranged into rounds of some given number s of simultaneous matches. This is an added restriction inasmuch as a player can now appear at most once in any of the matches within a round. The tournaments of this section can be arranged in rounds of N, the jth round containing the matches $M(1, j), \cdots, M(N, j)$.

3. Generators. Most of the designs which follow use a method of generators suggested by the technique of R. C. Bose [2] for designing incomplete balanced block experiments. For the moment, the men will be assumed numbered $0, 1, \cdots, n-1$ and likewise the women. A *generator* will be a particular match from which $n-1$ other matches may be derived. If the generator has the quadruple $G = (m_1, m_2, w_1, w_2)$ (interpreted as in Sec. 2), then for $t=1, \cdots, n-1$, other matches are

$$G(t) = (m_1 + t, m_2 + t, w_1 + t, w_2 + t),$$

where the addition is modulo n.

Now a list of g generators $G_1, \cdots, G_g$ will specify a tournament of ng matches $G_i(t)$ $i=1, \cdots, g$, $t=0, \cdots, n-1$. These ng matches will not satisfy

I or II, or even be distinct unless the generators G_i are chosen with care.

Table I below lists sets of generators for tournaments satisfying I for various values of n. For example when there are 8 men and 8 women the generators are (0102) and (0256). These generate a 16 match tournanent:

(0102), (1213), (2324), (3435), (4546), (5657), (6760), (7071),

(0256), (1367), (2470), (3501), (4612), (5723), (6034), (7145).

It remains to characterize the sets of generators which will produce tournaments satisfying I or II. To each generator $G=(m_1, m_2, w_1, w_2)$, associate four sets of numbers called MM differences, WW differences, WM_0 differences, WM_1 differences as follows.

MM differences are $m_2 - m_1$ and $m_1 - m_2$,

WW differences are $w_2 - w_1$ and $w_1 - w_2$,

WM_0 differences are $w_2 - m_1$ and $w_1 - m_2$,

WM_1 differences are $w_2 - m_2$ and $w_1 - m_1$.

These differences are to be computed modulo n. For example in the case of the two generators for $n=8$ the differences are listed below.

Generator	Differences			
	MM	WW	WM_0	WM_1
0102	± 1	± 2	2, 7	0, 1
0256	± 2	± 1	3, 6	4, 5

This set of differences satisfies the hypotheses of the following theorem.

Theorem I. *The tournament generated by g generators* $G_1, \cdots, G_g$ *satisfies* I *if and only if*

(i) *no two of the* $2gMM$ *differences are equal* modulo n,

(ii) *no two of the* $2gWW$ *differences are equal* modulo n,

(iii) *no two of the* $4g$ WM *differences are equal* modulo n.

Theorem II. *The tournament generated by g generators* $G_1, \cdots, G_g$ *satisfies* II *if and only if*

(i) *no two of the* $2g$ MM *differences are equal* modulo n,

(ii) *no two of the* $2g$ WW *differences are equal* modulo n,

(iii) *no two of the* $2g$ WM_0 *differences are equal* modulo n,

(iv) *no two of the* $2g$ WM_1 *differences are equal* modulo n.

The proofs of Theorems I and II are similar; only Theorem I will be proved in detail.

Man a and woman b play together twice in the tournament if and only if there are two generators (possibly the same) G and G^* such that

1. G contains a man m and a woman w and $a \equiv m+t$, $b \equiv w+t$ for some t.

2. G^* contains a man m^* and a woman w^* and $a \equiv m^* + t^*$, $b \equiv w^* + t^*$ for some t^*. Rewriting 1 and 2 as $w - m \equiv b - a \equiv w^* - m^*$, man a and woman b play together twice if and only if $b - a$ appears twice among the WM differences. Similarly the possibility of two men or two women meeting twice leads to the remaining conditions (i) and (ii) of Theorem I.

To achieve the upper bound (1) using Theorem I we must have $T = ng = [\frac{1}{2}n[\frac{1}{2}n]]$, which is possible only if $n \equiv 0$ or 1 (mod 4) and $g = [\frac{1}{4}n]$. For every such value of n in $5 \leqq n \leqq 17$ it has been found possible to find $[\frac{1}{4}n]$ generators and it seems reasonable to conjecture that even for larger $n \equiv 0$ or 1 (mod 4) the upper bound (1) can be achieved. Although Table I was easy to compute by cut and try methods a systematic way of producing $[\frac{1}{4}n]$ generators is known only when n is a power of an odd prime (next section). When $n \leqq 4$ it is impossible to find even a single generator satisfying the hypotheses of the theorem. When $n = 4$ there can be only two matches, such as (0101), (2323). When $n = 2$ or 3 only one match is possible.

When $n \equiv 2$ or 3 (mod 4) the method of generators can never produce a tournament of $[\frac{1}{2}n[\frac{1}{2}n]]$ matches. There is some evidence to show that this number of matches might still be had by other means. For example when $n = 6$ the upper bound is 9; a 9-match tournament is (0101), (0223), (0345), (1424), (1535), (2405), (2514), (3413), (3502). When $n = 7$ the upper bound is 10; a 10 match tournament is (0101), (0223), (0345), (1424), (1535), (2306), (2415), (3612), (4603), (5646). Unlike the tournaments of Section 2 or those constructed from generators, each player does not appear the same number of times in the 10 match tournament for $n = 7$; man 5 and woman 6 play only twice while the others play three times.

In any tournament of $[\frac{1}{2}n[\frac{1}{2}n]]$ matches satisfying I, and in which n is even, each man meets every woman. Similarly whenever the upper bound is achieved for a value of $n \equiv 1$ (mod 4) each man meets every woman but one. If the players consist of n married couples the tournament can then be arranged so that every man meets every woman except his own wife. For example, if the design is obtained from $[\frac{1}{4}n]$ generators there will be just one residue R (mod n) which does not appear among the WM differences. Then the men and women are to be numbered so that, for every $x = 0, \cdots, n-1$, man x is married to woman $x + R$ (mod n).

Table II lists sets of generators for tournaments satisfying II. For example when $n = 5$ the generators (0103), (0240) generate the tournament (0103), (1214), (2320), (3431), (4042), (0240), (1301), (2412), (3023), (4134). In this tournament man x plays against every other man and both with and against every woman except woman $x + 1$.

In a tennis match it may be a slight advantage to play on a particular side of the court (because of the position of the sun or the direction of wind). Suppose we always interpret the quadruple $(m_1 m_2 w_1 w_2)$ as a match in which m_1 and w_1 play on the better side of the court and m_2 and w_2 play on the other. Then each player appears g times on the better side and g times on the worse side in a tour-

nament constructed from g generators.

To generate a tournament by rounds of s, one may insist further that the generators $G_1, \cdots, G_g$ each have no MM or WW differences $0, \pm 1, \cdots$, or $\pm(s-1) \pmod n$. For suppose $G=(m_1, m_2, w_1, w_2)$ is a generator with such MM and WW differences. Consider the two matches $G(t)$ and $G(T)$, generated by G and containing a given player, say woman b. Then $b \equiv w_1+t \equiv w_2+T \pmod n$ and $T-t \equiv w_1-w_2 \pmod n$. Thus the player appears at most once in any collection of s consecutive matches of the sequence $G, G(1), \cdots, G(n-1)$. For $j=1, \cdots, [n/s]$, the set of s matches

$$R(j) = G(sj), \qquad G(sj+1), \cdots, G(sj+s-1)$$

is a round which can be played simultaneously. Thus the tournament provides $g[n/s]$ rounds of s. If s does not divide n, there will be $g(n-[n/s])$ other matches which can be played in g smaller rounds. If constraint I is adopted, the additional restriction on the MM and WW differences is not always severe. For example for $n=15$ and $s=5$ there are still three generators (0, 5, 0, 6)(0, 7, 9, 14), (0, 6, 4, 11).

TABLE I

SETS OF GENERATORS FOR TOURNAMENTS SATISFYING I

n	Generators
5, 6, or 7	(0 1 0 2)
8, 9, 10, or 11	(0 1 0 2), (0 2 5 6)
12	(0 1 0 4), (0 2 7 10), (0 4 1 6)
13	(0 1 0 2), (0 2 7 8), (0 5 3 9)
14 or 15	(0 1 0 4), (0 2 7 10), (0 4 1 6)
16, 19 and over	(0 1 0 6), (0 4 12 13), (0 6 1 10), (0 5 3 7)
17 and over	(0 1 0 2), (0 2 9 10), (0 8 6 11), (0 7 4 12)

TABLE II

SETS OF GENERATORS FOR TOURNAMENTS SATISFYING II

n	Generators
5	(0 1 0 3), (0 2 4 0)
6	(0 1 0 2), (0 2 5 4)
7	(0 1 0 2), (0 2 3 0), (0 3 6 5)
8 and over	(0 1 0 2), (0 2 3 6), (0 3 6 5)

4. Designs for n a power of an odd prime. This section will give a systematic procedure for designing tournaments satisfying I or II when $n=p^B$, p an odd prime, B some integer $\geqq 1$. The method of generators will be used. Since the cases $n=p^B$ with $B \geqq 2$ require a slight generalization of the results of Section 3, the simpler case $n=p$ will be given in detail first.

Every prime n has a primitive root (see Nagell [4]), *i.e.*, an integer a such that $a, a^2, \cdots, a^{n-1} \equiv 1$ are all different modulo n. Set $k = [\frac{1}{4}n]$ and construct a generator

$$G = (0, a^k - 1, a^k, -1). \tag{3}$$

Its differences are

$$\begin{aligned} MM \text{ differences} &= \pm (a^k - 1), \\ WW \text{ differences} &= \pm (a^k + 1), \\ WM_0 \text{ differences} &= \pm 1, \\ WM_t \text{ differences} &= \pm a^k. \end{aligned}$$

The other generators will be $G_1, G_2, \cdots$, where

$$G_i = a^i G = (0, a^i(a^k - 1), a^{i+k}, -a^i). \tag{4}$$

The differences of G_i are those of G multiplied by a^i.

If constraint II is adopted, the $\frac{1}{2}(n-1)$ generators $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ will be used. In Theorem II, each set of differences contains constant multiples of $\pm 1, \pm a, \cdots, \pm a^{\frac{1}{2}(n-3)}$. These differences are all distinct. For, if $a^i \equiv \pm a^j$, then $a^{i-j} \equiv \pm 1$, which implies that $\frac{1}{2}(n-1)$ divides $i-j$. Now Theorem II applies and $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ generate a tournament which satisfies II and achieves the bound (2).

If I is adopted, we use only $[\frac{1}{4}n]$ generators $G, G_1, \cdots, G_{k-1}$ in order to prevent equalities between WM_0 and WM_t differences. Theorem I applies and the tournament of $n[\frac{1}{4}n]$ matches satisfies I. If $n = 4k+1$, the bound (1) is achieved. The only man-woman pairs which do not meet are those with $m \equiv w \pmod{n}$.

Similar designs are obtained for $n = p^B$ (where $B \geqq 2$) by using a field of p^B elements. The names of the players are no longer integers modulo n but are elements of the field. Generators are defined as in Section 3 but with $+$ the addition operation of the field and t ranging over all p^B field elements. Theorems I and II still hold; indeed their proofs require only that $+$ be a commutative group operation.

Again a primitive root a exists such that $a, a^2, \cdots, a^{n-1} = 1$ are the nonzero field elements [3]. Set $k = [\frac{1}{4}n]$ and construct generators G, G_i as quadruples of field elements using (3) and (4). Again $G, G_1, \cdots, G_{\frac{1}{2}(n-3)}$ and $G, G_1, \cdots, G_{k-1}$ satisfy Theorems II and I.

For illustration take $n = 3^2$. A field of 9 elements consists of 9 polynomials $0, \pm 1, \pm x, \pm(x+1), \pm(x-1)$ with coefficients integers modulo 3. The product of two elements is found by multiplying the two polynomials in the natural way, dividing the result by a mod 3 irreducible polynomial $x^2 - x - 1$, and keeping only the remainder. In this field the polynomial x is a primitive root. In the design for constraint I, the two generators and their differences are listed below:

GENERATOR	MM	WW	WM_0	WM_1
$G=(0, x, x+1, -1)$	$\pm x$	$\pm(x-1)$	± 1	$\pm(1-x)$
$xG=(0, x+1, 1-x, -x)$	$\pm(x+1)$	± 1	$\pm x$	$\pm(1-x)$

The generalization fails if $n=2^B$. In a field of 2^B elements, $-d=d$ for all elements d; then the MM differences cannot be distinct.

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MATHEMATICAL NOTES

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THE SEPARATION OF PARTIAL DIFFERENTIAL EQUATIONS WITH MIXED DERIVATIVES

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The classical argument for the separation of variables in partial differential equations is often too weak to separate equations with mixed derivatives. We present a stronger separability argument which may also be used for partial differential equations with mixed derivatives.

To define separability, let $L[U]$ be a partial differential operator defined on a domain D of an n-dimensional Euclidean space. The partial differential equation $L[U]=0$ is said to be separable in the $x=(x_1, \cdots, x_n)$ variables if there exists a solution to $L[U]=0$ in the form

$$U(x) = \prod_{i=1}^{n} X^i(x_i), \tag{1}$$

where each X^i, $i=1, 2, \cdots, n$, is a function of x_i only.

A sufficient condition for separability is:

LEMMA 1. *Let $M^i[V_i]$, $i=1, \cdots, n$, denote partial differential operators on the functions $V^i(\bar{x}_i)$ defined on D with $\bar{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$,*

$i=1, \cdots, n$. Let $N^i[X^i(x_i)]$, $i=1, \cdots, n$, denote ordinary differential operators defined on D. Then the partial differential equation $L[U]=0$ is separable if there exist operators M^i and N^i such that solutions to the n ordinary differential equations $N^i[X^i]=0$ exist in D and such that

$$L[U] = \sum_{i=1}^{n} M^i[V^i]N^i[X^i]$$

for x in D.

The proof is obvious since the solution in the form of equation (1) can be constructed at once from solutions to the n ordinary differential equations $N^i[X^i]=0$.

In general, the above lemma yields the same solutions as the classical argument when the classical argument is applicable. Furthermore, it is more straightforward. For example, consider the biharmonic equation $\nabla^4 U(x, y)=0$. Here, we assume that $U(x, y) \equiv X(x)\,Y(y)$ and write

$$X^{(\mathrm{iv})}Y + 2X''Y'' + XY^{(\mathrm{iv})} \equiv [Y'' + Y(d^2/dx^2)](X'' + kX) + [X'' + X(d^2/dy^2)](Y'' - kY) = 0$$

where k is arbitrary. We may then use solutions to the ordinary differential equations $X''+kX=0$ and $Y''-kY=0$ to construct solutions to the biharmonic equations.

It is not difficult to find a non-orthogonal coordinate system in which Laplace's equation is separable [1]. For this, let

$$x = u, \qquad y = \tfrac{1}{2}(u - v), \qquad z = w - \tfrac{1}{4}\sqrt{2}(3u - v).$$

In the nonorthogonal u, v, w-coordinates,

$$\nabla^2 F \equiv F_{,uu} + 5F_{,vv} + 2F_{,ww} + 2F_{,uv} + \sqrt{2}F_{,uw} - \sqrt{2}F_{,vw}.$$

Now let $F(u, v, w) = U(u)\,V(v)\,W(w)$, and Laplace's equation may be written in the form

$$L(U' - kU) + M(V' + kV) + 2UV(W'' + \sqrt{2}kW' + 2k^2W) = 0$$

where k is arbitrary and

$$L = V'W + \sqrt{2}VW' + VW\frac{d}{du}, \qquad M = U'W - 4kUW - \sqrt{2}UW' + 5UW\frac{d}{dv}.$$

A solution to the partial differential equation may now be constructed from solutions to the ordinary differential equations

$$U' - kU = 0, \qquad V' + kV = 0, \quad \text{and} \quad W'' + \sqrt{2}kW' + 2k^2W = 0.$$

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ON AN OPTIMAL SEARCH PROCEDURE

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One of the n objects of a set S will obtain a certain property by a random choice such that the ith object has the probability P_i of being chosen ($\sum_{i=1}^{n} P_i = 1$). Afterwards the chosen object, called A, will be identified by the following test sequence. Assuming that after h tests ($h = 0, 1, \cdots$) A is found to belong to a subset of m_h objects ($m_0 = n$), the procedure is terminated if $m_h = 1$, otherwise the $(h+1)$th test consists in splitting this subset into two subsets and determining which of these contains A. Zimmerman [1] proved that the reversion of a combination procedure, such that in each stage the objects with the two smallest probabilities are combined into a new object and the sum of the two probabilities is assigned as the probability of this object, gives a test procedure with minimum expected number of tests.

In this note we shall consider the particular case when all P_i are equal. Writing $m_h = 2^{p_h} + k_h$, where p_h and k_h are nonnegative integers and $k_h < 2^{p_h}$ we shall prove the following

THEOREM. *When all P_i are equal the minimum expected number of tests is $p_0 + (2k_0/n)$, and this will be attained if and only if the number of objects of each of the two groups which are tested in the $(h+1)$*th *test ($h = 0, 1, \cdots$) belongs to the closed interval $(2^{p_h - 1}, 2^{p_h})$.*

Instead of using the combination device we base our proof directly on the splitting procedure.

Proof. The theorem holds for $n = 1$, since no testing is needed in this case. Assuming that the theorem holds for $1, \cdots, n-1$ objects, we shall show that it holds for n objects. We divide the set S into two subsets of n_1 and n_2 objects, with $n_1 \leqq n_2$. Dropping the subscripts of p_0 and k_0 we shall prove that for n objects the minimum number of tests equals $p + (2k/n)$, and that this minimum is attained if and only if n_1 and n_2 both belong to the closed interval $(2^{p-1}, 2^p)$ and subsequent testing follows the condition of the theorem.

We shall write $n_1 = 2^{p-1-r} + i$ and $n_2 = 2^{p-1+s} + j$, where $p-1-r$, $p-1+s$, r, s, i, j are nonnegative integers, $i < 2^{p-1-r}$ and $j < 2^{p-1+s}$. Further $p-1+s \leqq p$, so that s equals 0 or 1. Let $m(n_1, n_2)$ denote the minimum expected number of tests (n_1, n_2 fixed); $p(A, n_i)$, the probability that A belongs to the subset with n_i objects; and $\mu(A, n_i)$, the minimum expected number of tests given that A belongs to the subset with n_i objects. Then holding n_1, n_2 fixed, we have, since by assumption the theorem applies to $1, \cdots, n-1$ objects,

$$\begin{aligned} m(n_1, n_2) &= 1 + \sum_{i=1}^{2} p(A, n_i)\mu(A, n_i) \\ &= 1 + (n_1/n)[p - 1 - r + 2(i/n_1)] + (n_2/n)[p - 1 + s + 2(j/n_2)] \\ &= p + (2k/n) + R, \end{aligned}$$

where R, by means of the above expressions for n, n_1, n_2, can be written

$$R = (1/n)(n_2 s - n_1 r + 2^{p+1} - 2^{p-r} - 2^{p+s}).$$

Since s can take only the values 0 and 1 we can write

$$R = (1/n)(2^p - n_1 r - 2^{p-r} + a), \qquad a = \begin{cases} 0, & s = 0, \\ j, & s = 1. \end{cases}$$

Hence $R=0$ for (i) $r=0$, $s=0$, and for (ii) $r=0$, $s=1$, $j=0$, while $R>0$ for (iii) $r=0$, $s=1$, $j>0$, and, since $n_1<2^{p-r}$, also for the remaining case, (iv) $r>0$. Since n_1 and n_2 both belong to the closed interval $(2^{p-1}, 2^p)$ if and only if (i) or (ii) holds, and since, by assumption, any departure, in subsequent testing, from the condition of the theorem will add a positive quantity to the above expression for the minimum expected number of tests, the theorem follows.

The theorem can be applied to testing a system consisting of a set of n components and having the following properties. (i) While the system functions all of its components have the same probability of failing within a unit of time. (ii) Only one component fails at a time. (iii) The system breaks down as soon as one of its components fails. (iv) No further component will fail while the system is tested and repaired. (v) Any subset of i components $(1 \leqq i \leqq n)$ can be tested in a single test. (vi) The testing is error free.

REMARK. In [2], which deals with the different problem of searching for all A-objects in a set where initially each object independently of the others has the probability P of becoming an A-object, some limiting results are obtained, to which the above theorem is closely related (cf. formulas (20a), (21), and (43b) of [2]).

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ON SOLUTIONS OF HOMOGENEOUS, LINEAR, DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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THEOREM. (a) *Let*

$$y_k = \sum_{j=1}^{t} \left(\sum_{i=1}^{n_j} \beta_{ij} k^{i-1} \right) r_j^k + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i r_i^k, \qquad k = 0, 1, \cdots, \tag{1}$$

be the general solution of a homogeneous, linear, difference equation of order n with real, constant coefficients, where $n_j \geqq 1$, $j=1, \cdots, t$, *and* $(n_1+\cdots+n_t) \leqq n$. β_{ij} *and* α_i *are arbitrary constants, and* r_j, $j=1, \cdots, t$, *and* r_i, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$, *are distinct, real or complex numbers. Let m be a positive integer, and let* $p \neq 0$, $q \neq 0$ *be arbitrary real constants such that* $p+qr_j^m \neq 0$,

$j=1, 2, \cdots, t$, *and* $p+qr_i^m \neq 0$, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$. *If we define*

$$Y_{k,m} = \sum_{s=0}^{k} \binom{k}{s} p^{k-s} q^s y_{sm}, \tag{2}$$

then

$$Y_{k,m} = \sum_{j=1}^{t} \left\{ \beta_{1j} + \sum_{\nu=1}^{n_j-1} \gamma_{\nu j} k^\nu \right\} (p + qr_j^m)^k + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i (p + qr_i^m)^k, \tag{3}$$

where the $\gamma_{\nu j}$ *are arbitary constants. Thus,* $Y_{k,m}$ *is the general solution of some homogeneous, linear, difference equation of order* N *with real, constant coefficients, where* $N \leqq n$. *If the values of* r_j^m, $j=1, \cdots, t$, *and* r_i^m, $i=n, n-1, \cdots, (n_1+\cdots+n_t)+1$, *are distinct, then* $N=n$ *(for* $m=1$, *this is always so).*

(b) *In* (1), *let* $n_j=1$ *and* $\beta_{1j}=\alpha_j$, $j=1, \cdots, t$, *noting that* r_i, $i=1, \cdots, n$, *are distinct, real or complex numbers. If* $r_i^m \equiv C \neq 0$, $i=1, \cdots, n$, *where* C *is a real constant, and if* $m=\sigma n$, $\sigma=1, 2, \cdots$, *then* $N=1$, *and* $Y_{k,m=\sigma n} \equiv y_0(p+qC^\sigma)^k$ *is the general solution. If* (1) *(with* $n_j=1$, $j=1, \cdots, t$*) is the general solution of*

$$y_{k+n} = qy_{k+m} + py_k, \qquad pq \neq 0,\ 1 \leqq m < n, \tag{4}$$

then $Y_{k,m} \equiv y_{kn}$, $k=0, 1, \cdots$. *If* (1) *(with* $n_j=1$, $j=1, 2, \cdots, t$*) is the general solution of* (5),

$$py_k + qy_{k+n} = y_{k+1}, \qquad pq \neq 0, \tag{5}$$

then $Y_{k,n} \equiv y_{kn}$, $k=0, 1, \cdots$.

Remarks. A Fibonacci sequence $\{f_k\}$ is defined by $f_{k+2}=f_{k+1}+f_k$, $k \geqq 0$, where f_0 and f_1 are two arbitrary positive integers. If we set $p=1$, $q=1$, $m=1$, and $n=2$ in (4) and (2), we obtain

$$f_{2k} = \sum_{s=0}^{k} \binom{k}{s} f_s. \tag{6}$$

(6) solves the elementary problem E1347 [1959, 61] of this MONTHLY, vol. 66, p. 592. If we set $p=-1$, $q=1$, and $n=2$ in (5) and (2), we obtain

$$f_k = (-1)^k \sum_{s=0}^{k} (-1)^s \binom{k}{s} f_{2s}. \tag{7}$$

(7) is of interest when k is odd. In Jordan ([1], p. 132), we find the summation formula,

$$\sum_{s=0}^{k} (-1)^s \binom{k}{s} g(s) = (-1)^k \Delta^k g(0). \tag{8}$$

Thus, (8) also yields (7) if $g(s) \equiv f_{2s}$, since $\Delta^k f_{2s} = f_{2s+k}$. If $g(s) \equiv f_{2s+1}$, then (8)

yields

$$f_{k+1} = (-1)^k \sum_{s=0}^{k} (-1)^s \binom{k}{s} f_{2s+1}. \tag{9}$$

(9) is of interest when k is odd. The paper by Rao [2] has an extensive list of identities satisfied by the Fibonacci numbers, but fails to list either (7) or (9) which, apparently, are not too well known.

Proof of the theorem. (a) Substitution of y_{sm}, given by (1), into (2), yields, upon interchanging summations,

$$Y_{k,m} = \sum_{j=1}^{t} \sum_{i=1}^{n_j} \beta_{ij} m^{i-1} \sum_{s=0}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} + \sum_{i=(n_1+\cdots+n_t)+1}^{n} \alpha_i (p + qr_i^m)^k.$$

Now,

$$\begin{aligned} A_j &\equiv \sum_{i=1}^{n_j} \beta_{ij} m^{i-1} \sum_{s=0}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} \\ &= \beta_{1j}(p + qr_j^m)^k + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{s=1}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} \\ &= (p + qr_j^m)^k \left[\beta_{1j} + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{s=1}^{i-1} \mathfrak{S}_{i-1}^s (qr_j^m)^s (p + qr_j^m)^{-s} k^{(s)} \right], \end{aligned}$$

since

$$\sum_{s=1}^{k} \binom{k}{s} p^{k-s} (qr_j^m)^s s^{i-1} = (p + qr_j^m)^k \sum_{s=1}^{i-1} \mathfrak{S}_{i-1}^s (qr_j^m)^s (p + qr_j^m)^{-s} k^{(s)}. \tag{10}$$

(10) follows from ([1], pp. 196–197, formulae (2) and (7), where $u(t) = (p+qr_j^m t)^k$ in (7)). In (10), $\mathfrak{S}_{i-1}^s$ is a Stirling number of the second kind, *i.e.*, $k^i = \sum_{s=1}^{i} \mathfrak{S}_i^s k^{(s)}$, where $k^{(s)} = k(k-1)\cdots(k-s+1) = \sum_{\nu=1}^{s} S_s^\nu k^\nu$ and S_ν^s is a Stirling number of the first kind. Let

$$h(s, i) = \mathfrak{S}_{i-1}^s (qr_j^m)^s (p + qr_j^m)^{-s}.$$

Then

$$\sum_{s=1}^{i-1} h(s, i) \sum_{\nu=1}^{s} S_\nu^s k^\nu = \sum_{\nu=1}^{i-1} \left(\sum_{s=\nu}^{i-1} h(s, i) S_\nu^s \right) k^\nu.$$

Let $H(i, \nu) = \sum_{s=\nu}^{i-1} h(s, i) S_\nu^s$. Then

$$\begin{aligned} (p + qr_j^m)^{-k} A_j &= \beta_{1j} + \sum_{i=2}^{n_j} \beta_{ij} m^{i-1} \sum_{\nu=1}^{i-1} H(i, \nu) k^\nu \\ &= \beta_{ij} + \sum_{\nu=1}^{n_j-1} \left(\sum_{i=\nu}^{n_j-1} m^i H(i+1, \nu) \beta_{(i+1)j} \right) k^\nu = \beta_{1j} + \sum_{\nu=1}^{n_j-1} \gamma_{\nu j} k^\nu, \end{aligned}$$

where

$$(11)\quad \gamma_{vj} = \sum_{i=v}^{n_j-1} m^i H(i+1, v)\beta_{(i+1)j} = \sum_{i=v}^{n_j-1} \sum_{s=v}^{i} m^i S_v^s \mathfrak{S}_i^s (qr_j^m)^s (p + qr_j^m)^{-s} \beta_{(i+1)j}.$$

(b) If $n_j=1$ and $\beta_{1j}=\alpha_j$, $j=1, \cdots, t$, then $y_k=\sum_{i=1}^{n}\alpha_i r_i^k$ and $Y_{k,m}=\sum_{i=1}^{n}\alpha_i(p+qr_i^m)^k$. If y_k is the general solution of (4), then $p+qr_i^m=r_i^n$, $i=1, \cdots, n$. If y_k is the general solution of (5), then $p+qr_i^n=r_i$, $i=1, \cdots, n$.

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ON THE ORDER OF A BIAS

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A quantity which often arises in statistics is u/v where $u=(\bar{x}-\mu_x)^2$, $v=\sum_{i=1}^{n}(x_i-\bar{x})^2$, and $\bar{x}$ is the mean of a sample $(x_1, \cdots, x_n)$ drawn from a population with mean μ_x. The form arises, for example, in evaluating the variance of the linear regression estimator of a population mean, see Cochran [1]. It has been shown ([3], [4]) that u and v are independently distributed if and only if the x distribution is Gaussian. Much literature has been devoted to the effects of non-normality on u/v, most of which is directed at the distribution properties and analysis of Type I and Type II errors. In general $E(u/v)=E(1/v)Eu+\mathrm{Cov}(u, 1/v)$, where E denotes mathematical expectation and Cov denotes the covariance of two random variables. If the higher moments are finite, it can be shown by an interesting straightforward method that the covariance term is of lower order in n than $E(1/v)Eu$.

If u/v is written as $U(1+\delta_u)(1+\delta_v)^{-1}/V$ and $1/v$ as $(1+\delta_v)^{-1}/V$ where $Eu=U, Ev=V\neq 0$, $\delta_u=(u-U)/U$ and $\delta_v=(v-V)/V$ then $\mathrm{Cov}(u, 1/v)=(U/V)E\{\delta_u(1+\delta_v)^{-1}\}\doteq -\mathrm{Cov}(u, v)/V^2$. The approximation involves dropping the terms $UE\{\delta_u\delta_v^{2+i}\}/V$, $i=0, 1, \cdots$. The terms inside the expectation cannot be greater than $O(n^{-1})$ by the methods of Fisher [2]. Further, $U=\sigma_x^2/n$ and $V=(n-1)\sigma_x^2$ so that the neglected terms are no more than $O(n^{-3})$.

Next, let $d_i=x_i-\mu_x$ so that $Ed_i=0$ and $Ed_i^2=\sigma_x^2$, then

$$\mathrm{Cov}(u, v)=\frac{1}{n^3}E\left\{\left[(n-1)\sum_{i=1}^{n}d_i^2-\sum_{i\neq j}^{n}d_id_j\right]\left[\sum_{i=1}^{n}d_i^2+\sum_{i\neq j}^{n}d_id_j\right]\right\}-\frac{n-1}{n}\sigma_x^4.$$

which will reduce to

$$(1)\qquad \frac{n-1}{n^2}\{Ed_i^4-3\sigma_x^4\}$$

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or alternatively, since by definition the variance of d_i^2 is $\mathrm{Var}(d_i^2) = Ed_i^4 - \sigma_x^4$,

$$\frac{n-1}{n^2}\left\{\mathrm{Var}(d_i^2) - 2\sigma_x^4\right\}. \tag{2}$$

Hence, $\mathrm{Cov}(u, v)$ is $O(n^{-1})$ and $\mathrm{Cov}(u, 1/v)$ is $O(n^{-3})$. Furthermore, Eu is $O(n^{-1})$ and $E(1/v)$ is $O(n^{-1})$ so that $EuE(1/v)$ is $O(n^{-2})$. Notice the interesting agreement of (1) and (2) with the known fact that $\mathrm{Cov}(u, v) = 0$ for normal populations. The straightforward approach used to derive these expressions can also be used to verify the stated orders of the terms neglected in the approximation.

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AN ELEMENTARY ANALYSIS OF AN INTEGRAL QUADRATIC FORM

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The quadratic form,

$$x^2 + hxy + ky^2 \tag{1}$$

in integers x and y, with integral coefficients, can be written as the determinant of the matrix,

$$(xI + yA), \tag{2}$$

where I is the two by two identity matrix and A is a two by two matrix with a trace and determinant of h and k respectively. The corresponding matrix form (2) defines a commutative ring with identity over the integers and an integral domain if A has no rational characteristic root. We shall obtain some general and particular results concerning binary integral quadratic forms with a leading coefficient of one by analysis of the ring defined by (2).

For a two by two matrix, H, let $|H|$, $\overline{H}$, and $\mathrm{tr}(H)$ denote, respectively, the determinant, the adjoint, and the trace of H. For a positive integer, n, and square matrices of integers U and V of like order, let the relation, $U \equiv V \pmod{n}$, mean that n divides each element of $U - V$. Let "diagonal matrix" mean "square matrix whose off-diagonal elements are zero." Let "scalar matrix" mean "diagonal matrix with each diagonal element equal to the same number."

The following relations are readily verified for two by two matrices, $A, B, K, \cdots$.

1.1 $\overline{uA+vI}=u\overline{A}+vI$, where u and v are numbers.
1.2 $A+\overline{A}=(\operatorname{tr} A)I$
1.3 $(A-\overline{A})^2=I[\operatorname{tr}(A^2)-2|A|\,]=-|A-\overline{A}|\,I$
1.4 $|A|=|\overline{A}|$ and $A\overline{A}=|A|\,I$
1.5 $|tI-A|=t^2-t(\operatorname{tr} A)+|A|=\frac{1}{4}|(2t-\operatorname{tr} A)I-(A-\overline{A})|$ $=\frac{1}{4}[(2t-\operatorname{tr} A)^2+|A-\overline{A}|\,]$, for each number t.
1.6 $\overline{AB}=\overline{BA}$
1.7 $(A-\overline{A})(K-\overline{K})$ is a scalar matrix if K is of the form $(xI+yA)$, for numbers x and y.
1.8 If $AB=BA$ and B is a diagonal matrix, then A is a diagonal matrix or B is a scalar matrix.

It should be emphasized that these statements, with the exception of (1.6), are valid only for two by two matrices.

We now introduce the following two lemmas:

LEMMA 1. *If U and V are two by two nondiagonal matrices of integers which commute, then there exist integers a, b, and c, where $ab\neq 0$, and $(a, b, c)=1$, such that $aU=bV+cI$.*

LEMMA 2. *If U and V are two by two nondiagonal matrices of integers which commute, and p is a prime integer which divides $|U|$ and $|V|$, then $UV\equiv 0 \pmod p$ or $\overline{U}V\equiv 0 \pmod p$.*

Proof of Lemma 1. It will suffice to show that there exists a rational linear combination of U and V which is a scalar matrix. Denote the elements of U and V by U_{ij} and V_{ij}, respectively, where the first index denotes the row position and the second, the column position.

The equation

$$\begin{pmatrix} U_{12} & V_{12} \\ U_{21} & V_{21} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has a rational nontrivial solution since, owing to the commutativity of U and V, $U_{12}V_{21}-V_{12}U_{21}=0$. Hence, $xU+yV$ is a diagonal matrix and commutes with U and V. Further, by (1.8) above, $xU+yV$ is a scalar, since U and V are not diagonal matrices. Neither xU nor yV is a scalar; hence, neither x nor y is zero.

Proof of Lemma 2. By Lemma 1, and relations (1.1), (1.2), and (1.4) above, there exist integers, a, b, and c, where $ab\neq 0$, and $(a, b, c)=1$, such that:
2.1 $aU=bV+cI=-b\overline{V}+(c+b\ \operatorname{tr} V)I$ and
2.2 $a\overline{U}=b\overline{V}+cI$.
Hence, $aUV=-b|V|\,I+(c+b\ \operatorname{tr} V)V$ and $a\overline{U}V=b|V|\,I+cV$. Since, by (1.4), (2.1), and (2.2), $a^2|U|\,I=a^2U\overline{U}=[b^2|V|+c(c+b\ \operatorname{tr} V)]I$, it follows that p divides c or p divides $(c+b\ \operatorname{tr} V)$. Therefore $aUV\equiv 0 \pmod p$ or $a\overline{U}V\equiv 0 \pmod p$. Suppose p divides a and c; then it cannot divide b since $(a, b, c)=1$ and hence divides $\overline{V}$. This establishes the lemma for, if p divides a and c (or a and $c+b\ \operatorname{tr} V$), then $V\equiv\overline{V}\equiv 0 \pmod p$.).

Suppose now that A is a two by two matrix of integers, with $h=\operatorname{tr} A$, $k=|A|$, and $D=|A-\overline{A}|$. Impose the restrictions that $-|A-\overline{A}|$ is not the square of an integer. By (1.5) above, this implies that A has no rational characteristic root. Let S denote the set of matrices $\{xI+yA\}$, M the set of integers $\{|uI+vA|\}$, where x, y, u, and v are integers, with $(u, v)=1$. Let P denote the set of positive prime factors of integers in M, and let P' denote the set of odd primes in P. A matrix U in S will be called an *S-unit* if $|U|^2=1$, an *S-composite* if it is the product of two nonunit matrices of S, and an *S-prime* if it is neither an S-unit nor an S-composite. It follows that S is an integral domain and is, in fact, isomorphic to the quadratic integral domain defined by the characteristic root field of A.

We now state and prove the following theorems:

THEOREM 1. *If p belongs to P and to M, and u is an integer such that pu belongs to M, then u belongs to M.*

THEOREM 2. *If to each p in P' there corresponds an odd integer m in M, less than p^2, such that p divides m, then P' is a subset of M. If P' is a subset of M, and either M contains no even integer or M contains an even integer not divisible by 4, then P is a subset of M.*

THEOREM 3. *If U is an S-prime and P is a subset of M, then $|U|$ is a prime integer.*

THEOREM 4. *If P is a subset of M, then S has unique factorability, up to an S-unit.*

Proof of Theorem 1. By Lemma 2, $(UV)/p$ or $(\overline{U}V)/p$ is in S. This proves the theorem since $|(UV)/p|=|(\overline{U}V)/p|=u$.

Proof of Theorem 2. Suppose P' is not a subset of M. Then let p be the least prime in P' which is not in M. But, by hypothesis, p divides an odd integer in M with a resulting odd quotient less than p. Thus each prime factor of this quotient is less than p, and hence, is in M. But by Theorem 1, it would follow that p is in M. Further, if P is a subset of M, and M contains no even integer, then P' is P; if M contains an even integer not divisible by 4, then M contains 2, since all odd prime factors of this integer are in P'.

Proof of Theorem 3. Suppose $|U|$ is not a prime and not a unit. Then, since P is a subset of M, there exist matrices B and C in S such that $|U|=|B||C|$, where $|B|$ is a prime p, and $|C|^2>1$. Then

$$U=\frac{UB\overline{B}}{p}=\frac{U\overline{B}}{p}B=\frac{UB}{p}\overline{B}.$$

One of these two forms gives U as the product of two matrices in S, neither of which has determinant ± 1.

Proof of Theorem 4. Let $\{U_i\}$, $1 \leqq i \leqq n_1$, and $\{V_i\}$, $1 \leqq i \leqq n_2$, be sets of S-primes such that $\prod_{i=1}^{n_1} U_i = \prod_{i=1}^{n_2} V_i$ and hence such that $|\prod_{i=1}^{n_1} U_i| = |\prod_{i=1}^{n_2} V_i|$. Thus, by Theorem 3 and the division algorithm for integers, for each i from 1 to n_1 and some j from 1 to n_2, it follows that $|U_i| = |V_j|$. Hence $n_1 = n_2$.

Suppose one of the U's, say U_1, is not the product of an S-unit and one of the V_i. But, $U_1\overline{U}_1 = |U|\,I = |V_j|\,I = V_j\overline{V}_j$ for some integer j from 1 to n_1. Then

$$U_1 = \left(\frac{\overline{V}_j U_1}{|U_1|}\right) V_j = \left(\frac{V_j U_1}{|U_1|}\right)\overline{V}_j$$

and $(\overline{V}_j U_1)/|U_1|$ or $(V_j U_1)/|U_1|$ is an S-unit. But from the assumption on U_1, $(\overline{V}_j U_1)/|U_1|$ cannot be an S-unit. Thus for some S-unit T, $U_1 = T\overline{V}_j$. If $\overline{V}_j$ is not one of the V_i, then $T|V_j|\,I\prod U_i = V_j^2\prod V_i$, where in the first product, $2 \leqq i \leqq n_1$, and in the second, $1 \leqq i \leqq n_1$, $i \neq j$. Since $|V_j|$ is a prime, then for one of the V's, say V_k, and some S-unit Z, $V_jV_k = V_j\overline{V}_jZ$. Hence $V_k = \overline{V}_jZ$, which contradicts the assumption that none of the V_i's is $\overline{V}_j$. This completes the proof.

We now proceed to state criteria which will aid in determining the presence of unique factorability.

It is evident that by a reversible linear transformation over the integers the form (1) can be reduced to one of the forms:

$$u^2 + uv + nv^2, \tag{3}$$

$$u^2 + nv^2, \tag{4}$$

where (3) is obtainable if h is odd and (4) if h is even.

Clearly, if $n > 0$, unity is the only number in M which is less than n, and n is in M. If, in addition, P' is a subset of M and n has an odd prime divisor, p, then $p = n$ since p belongs to P'; therefore to M; hence is not less than n. Therefore, with $n > 0$ and P' a subset of M, n is an odd prime or has the form 2^j for some positive integer j. Now, M, defined by (3), contains $n+2$ and $4n-1$, and M, defined by (4), contains $n+1$, $n+4$, and $n+9$. Then if n is positive and even, M contains $2(2^{j-1}+1)$ if defined by (3), and $4(2^{j-2}+1)$ if defined by (4). Since $2^{j-1}+1$ is odd unless $j=1$ and $2^{j-2}+1$ is an odd integer unless $j=1$ or 2, it follows that if n is even, P' is not a subset of M, if M is defined by (3) and $n > 2$, or if M is defined by (4) and $n > 4$.

Again, if n is odd and positive and M is defined by (4), P' is not a subset of M if $n+1$ has an odd prime divisor, for such a divisor would be less than n, hence not in M. If $n+1$ has the form 2^j, for some positive integer j, then 2^j+8 is in M, defined by (4), so that if $j > 3$, the integer $2^{j-3}+1 < n$ and has an odd prime divisor which is in P'. Therefore P' is not a subset of M defined by (4) if n is odd, positive, and distinct from each of the values: 1, 3, 7.

Now if p is in P' (that is, if p is an odd prime divisor of a member of M) and if $(n, p) = 1$, then there exist positive integers r, s, and t, where s is odd, t even, $2r \leqq p-1$, $s \leqq p-2$, and $t \leqq p-1$, such that if M is defined by (3), p

divides $\frac{1}{4}[(2r+1)^2+D]$ and $(s+1)^2+D$, each of which is in M; and if M is defined by (4), then p divides t^2+n and s^2+n, each of which is in M.

Now from the restrictions given on r, s, and t, we obtain:

3.1 $0<\frac{1}{4}[(2r+1)^2+D]<\frac{1}{4}[p^2+D]<p^2$, if $0<D<3p^2$

3.2 $0<(s+1)^2+D\leqq p^2-[2p-(D+1)]<p^2$, if $0<D<2p-1$

3.3 $0<t^2+n\leqq p^2-[2p-(n+1)]<p^2$, if $0<n<2p-1$

3.4 $0<s^2+n\leqq p^2-[4p-(n+4)]<p^2$, if $0<n<4(p-1)$.

From these inequalities, the above limitations on n, and Theorem 2, together with the observation that p belongs to P' only if $-D$ is a quadratic residue of p, we have the following conclusions:

4.1 2 is the only even positive value of n for which P' is a subset of M, defined by (3), in which case P is a subset of M.

4.2 With M defined by (4) the only positive values of n for which P' is a subset of M are 1, 2, 3, 4, and 7, with P a subset of M only if $n=1$ or 2.

4.3 If n is positive and odd and M is defined by (3), $P'=P$ and P is a subset of M only if there exists an odd prime q, with $D<3q^2$, such that $-D$ is not a quadratic residue of any odd prime less than q, in which case n is the least member of P and M contains no composite integer below n^2.

As an illustration, let $n=41$ for form (3). Then $D=163$, and -163 is seen to be a quadratic nonresidue of 3, 5, and 7, while $163<3(11)^2$. Hence, x^2+x+41 is a prime for $x=0, 1, 2, \cdots, 39$. Note that the elimination of merely 3, 5, and 7 here automatically eliminates from P all primes below 41.

Some positive values of D for which similar results are seen to occur are 3, 7, 11, 19, 43, 67, and 163; and the corresponding n values are 1, 2, 3, 5, 11, 17, and 41. Hardy and Wright list these special values of D and indicate that there is at most one more positive integer with this remarkable property.* In view of the exacting conditions required of D, it seems unlikely that there exists another such positive value.

Consider now form (3), where $n<0$. Then $D<0$. If p is in P' there exists an even integer s and an odd integer t, $s^2<p^2$, $t^2<p^2$, such that p divides s^2+D and t^2+D, and either s^2+D or $\frac{1}{4}(t^2+D)$ is an odd integer in M, according as n is even or odd, respectively. If this odd integer is positive, it clearly is less than p^2. Therefore, P' is a subset of M only if each prime factor of x^2+D is in M, for every integer x with $x^2+D<0$ and $x+n$ even.

As a conclusion, we note that if P' is a subset of M and P is not a subset of M, then S has unique factorability up to an S-composite whose determinant is 2^j, for some integer $j\geqq 2$, and that with M defined by 4, and $n\equiv -1 \pmod 4$, the corresponding S set is a subset of the S set for which M is defined by (3) and $\frac{1}{4}n+\frac{1}{4}$ the coefficient of y^2. Should the former lack unique factorability,

* G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1954, p. 213.

certain difficulties are surmounted if the latter possesses this desired property. As an instance of this, the general integral solution of the equation, $x^2+11y^2=z^n$, n a positive integer, can be obtained by regarding the form $u^2+uv+3v^2$, of which the form x^2+11y^2 is a special case.

ON A THEOREM OF M. EIDELHEIT CONCERNING RINGS OF CONTINUOUS FUNCTIONS

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It is the intention of this paper to give a simplified proof of the following theorem of M. Eidelheit [1]:

THEOREM. *Let X be a compact metrizable space and let $C(X)$ be the ring of all real-valued continuous functions on X endowed with the supremum norm. Let P be a closed subring of $C(X)$ containing the constant functions. Then P is isometrically isomorphic to the ring $C(Y)$ of all real-valued continuous functions on a compact metrizable space Y.*

Our proof will be a simple application of the Stone-Weierstrass theorem [3] and the following result [2]:

PROPOSITION. *Let X be a compact Hausdorff space. In order that X is metrizable, it is necessary and sufficient that $C(X)$ is separable.*

Proof of the theorem. In X we define a binary relation R as follows: for x and x' in X, we define xRx' to mean that $f(x)=f(x')$ for all functions f in P. The relation R is evidently an equivalence relation. Let Y be the quotient space X/R. Since the set $\{(x,x'): xRx'\}$ is closed in the product space $X\times X$, Y is Hausdorff. Also since X is compact and since the partition map is continuous, Y is compact.

Let f be a function in P and let y be a point in Y. If x and x' are representatives of the equivalence class y, then $f(x)=f(x')$. Hence there is no ambiguity if we define $(Tf)(y)=f(x)$. The mapping T is then a linear isometry of P into $C(Y)$. By our construction of Y, $T(P)$ separates points. As P is uniformly closed, we see that $T(P)$ is the whole of $C(Y)$.

We still have to show that the space Y is metrizable. By assumption, X is metrizable. Hence by the Proposition, $C(X)$ is separable. As $C(X)$ is both metrizable and separable, its subspace P is separable. Thus $T(P)=C(Y)$ is also separable. Therefore, by our Proposition again, Y is metrizable. This finishes the proof.

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A SIMPLE PROOF OF A KNOWN RESULT IN PARTITIONS

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Let $a_1, \cdots, a_k$ be a set of different positive integers, let a be their least common multiple and let $p(n)$ be the number of partitions of n into parts chosen from this set, *i.e.*, $p(n)$ is the number of solutions of the Diophantine equation

$$n = m_1a_1 + m_2a_2 + \cdots + m_ka_k,$$

where the m_i are nonnegative integers. We give here a very simple proof of the

THEOREM. $p(n) = \sum_{t=1}^{k} c_{kt}(n)n^{k-t}$, *where* $c_{kt}(n)$ *depends on the residue of* $n \pmod a$ *but not otherwise on* n.

The theorem is not new, for it is implicit in the results of Sylvester and Glaisher, who investigated the properties of $p(n)$ at great length [1, 3].

Recently Rieger [2] gave an entirely different proof of a similar result for the particular case in which $a_j=j$. His result was slightly weaker, as the modulus a was replaced by $k!$

If we put $p(0)=1$ and take $|x|<1$, we have

$$F(x) = \prod_{j=1}^{k} (1 - x^{a_j})^{-1} = \sum_{n=0}^{\infty} p(n)x^n.$$

But $(1-x^a)(1-x^{a_j})^{-1}$ is a polynomial in x of degree $a-a_j$ and so

$$\sum_{n=0}^{\infty} p(n)x^n = P(x)(1 - x^a)^{-k} = P(x) \sum_{h=0}^{\infty} Q(h)x^{ah},$$

where $P(x) = \sum_{r=0}^{R} b_rx^r$ is a polynomial in x of degree $R = \sum_{j=1}^{k} (a-a_j) < ka$ and $Q(h) = (h+1)(h+2) \cdots (h+k-1)/(k-1)!$ is a polynomial in h of degree $k-1$. Since $Q(h)=0$ for $h=-1, -2, \cdots, -(k-1)$, we have

$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{r=0}^{R} b_rx^r \sum_{h=-(k-1)}^{\infty} Q(h)x^{ah}.$$

Hence, for $n>0 \geqq R-ka$, we have

$$p(n) = \sum_{r \equiv n \pmod a} b_rQ(\{n - r\}/a) = \sum_{t=1}^{k} c_{kt}(n)n^{k-t},$$

where

$$c_{kt}(n) = a^{t-k}\{(k - t)!\}^{-1} \sum_{r \equiv n \pmod a} b_rQ^{(k-t)}(-r/a).$$

Thus $c_{kt}(n)$ depends on the residue of $n \pmod a$ but not otherwise on n.

Much more information about $p(n)$ can, of course, be found by expressing $F(x)$ in partial fractions. This is essentially the method used by Sylvester and Glaisher.

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THE CONGRUENCE $(p-1/2)! \equiv \pm 1 \pmod{p}$

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Let p be an odd prime. Then Wilson's classical result states that $(p-1)!+1 \equiv 0 \pmod{p}$. On noting that $p-r \equiv -r \pmod{p}$, this gives, when $p \equiv 1 \pmod 4$, as is well known,

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 + 1 \equiv 0 \pmod{p}.$$

However, when $p \equiv 3 \pmod 4$, we have

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 - 1 \equiv 0 \pmod{p}.$$

Hence

$$\left(\frac{p-1}{2}\right)! \equiv (-1)^a \pmod{p}, \tag{1}$$

where $a = 0$ or 1. In view of the history of the question, it may perhaps be worth while to state and prove the

THEOREM.* *If p is a prime $\equiv 3 \pmod 4$ and $p > 3$, then in* (1)

$$a \equiv \tfrac{1}{2}[1 + h(-p)] \pmod 2 \tag{2}$$

where $h(-p)$ is the class number of the quadratic field $k\{\sqrt{(-p)}\}$.

This result does not appear to have been explicitly stated or at any rate does not seem well known. It is, however, implicit in the literature, and it is now a trivial deduction from results long known, *e.g.*, an old one of Dirichlet's (1828) given here as (3). In fact, Jacobi (1832) conjectured a result equivalent to (2) at a time when the class-number formula was not known. For the history of the subject, see Dickson's *History of the Theory of Numbers*, Vol. 1, page 275.

Write $E = [\frac{1}{2}(p-1)]!$ Denote by $r_1, r_2, \cdots$ the R quadratic residues of p less than $\frac{1}{2}p$, and by $n_1, n_2, \cdots$ the N quadratic nonresidues less than $\frac{1}{2}p$. Then the quadratic residues $r_1', r_2', \cdots$ greater than $\frac{1}{2}p$ are given by $p-n_1, p-n_2, \cdots$ since $p \equiv 3 \pmod 4$. Then

$$E = r_1 r_2 \cdots n_1 n_2 \cdots \equiv (-1)^N r_1 r_2 \cdots r_1' r_2' \cdots \equiv (-1)^N \bmod p \text{ if } p > 3, \tag{3}$$

* Professor Chowla informs me that he found the result about the same time that I did.

since $(-1)^N E \equiv g^{2+4+\cdots+(p-1)} = g^{\frac{1}{4}(p^2-1)} = (g^{\frac{1}{4}(p+1)})^{\frac{1}{2}(p+1)} \equiv 1 \pmod{p}$, where g is a primitive root of p.

Now $R+N=\frac{1}{2}(p-1)$, and it is known* from the class-number formula that

$$R - N = \delta h(-p), \qquad \begin{cases} \delta = 1 \text{ if } p \equiv 7 \pmod 8, \\ \delta = 3 \text{ if } p \equiv 3 \pmod 8,\ p > 3. \end{cases}$$

Hence $2N=\frac{1}{2}(p-1)-\delta h(-p)$. Then if $p\equiv 7 \pmod 8$, $2N\equiv 3-h(-p) \pmod 4$, and if $p\equiv 3 \pmod 8$, $2N\equiv 1-3h(-p) \pmod 4$. The first is $N\equiv\frac{1}{2}[-1-h(-p)]$ (mod 2) and the second is $N\equiv\frac{1}{2}[1+h(-p)]$ (mod 2). These are both included in $N\equiv\frac{1}{2}[1+h(-p)]$ (mod 2).

* L. Holzer, Zahlentheorie II, 1959, Leipzig, pp. 91–93, and H. Hasse, Vorlesungen über Zahlentheorie, 1950, Berlin, pp. 386–390.

A GENERALIZED TURÁN EXPRESSION FOR THE BESSEL FUNCTIONS

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1. In a recent paper Toscano [3] has proved the formula

$$\sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} H_{n+r}(x) H_{n-r}(x) = \frac{(2m)!(n-m)!}{m!} \sum_{j=m}^{n} \binom{j-1}{m-1} \frac{H_{n-j}^2(x)}{(n-j)!} \qquad (m \leqq n) \tag{1.1}$$

where $H_n(x)$ is the Hermite polynomial of order n. The expression in the left hand side may be regarded as a generalization of the Turán expression $H_n^2(x) - H_{n+1}(x)H_{n-1}(x)$. Indeed (1.1) reduces to the Demir-Hsü formula [2] when $m=1$. Other proofs of (1.1) as well as extensions to the Laguerre and ultraspherical polynomials and other hypergeometric functions are given in [1].

In the present note we obtain a similar formula involving the Bessel functions. We prove

$$\Omega_n^{(m)}(x) = \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} J_{n-r}(x) J_{n+r}(x) = \frac{4^m(2m)!}{x^{2m} m!(m-1)!} \sum_{k=0}^{\infty} (n+m+2k)(k+1)_{m-1}(n+k-1)_{m-1} J_{n+m+2k}^2(x) \tag{1.2}$$

where $(a)_m = a(a+1)(a+2)\cdots(a+m-1)$, $(a)_0=1$. For definition of the Bessel function $J_n(x)$ see [4]. This formula reduces, for $m=1$, to Lommel's formula [4, p. 152]

$$\tfrac{1}{4}x^2\{J_n^2(x) - J_{n+1}(x)J_{n-1}(x)\} = \tfrac{1}{4}x^2\Delta_n(x) = \sum_{k=0}^{\infty} (n+1+2k) J_{n+1+2k}^2(x). \tag{1.3}$$

It is also a positive representation as sum of squares.

We shall also discuss the cases $m=2$; $m=2$, $n=0$; and $n=0$ of (1.2).

2. *Proof of formula* (1.2). We first note that the formula

$$\sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} \cos 2r\theta = 2^{2m} \sin^{2m}\theta \tag{2.1}$$

can be proved very easily by substituting $\frac{1}{2}(e^{2r\theta i}+e^{-2r\theta i})$ for $\cos 2r\theta$ in the left hand side of (2.1) and summing the resulting two series separately.

It now follows from the formula [4, p. 150]

$$J_\mu(x)J_\nu(x) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} J_{\mu+\nu}(2x\cos\theta)\cos(\mu-\nu)\theta d\theta \tag{2.2}$$

and (2.1) that

$$\Omega_n^{(m)}(x) = \frac{2^{2m+1}}{\pi}\int_0^{\frac{1}{2}\pi} J_{2n}(2x\cos\theta)\sin^{2m}\theta d\theta \tag{2.3}$$

This formula can also be written in the form

$$x^{2m}\Omega_n^{(m)}(x) = \frac{2}{\pi}\int_0^{2x} J_{2n}(y)\{4x^2-y^2\}^{m-\frac{1}{2}}dy. \tag{2.4}$$

Put $\phi_m(x)=x^{2m}\Omega_n^{(m)}(x)$. Then $\phi_m(0)=0$ $(m\neq 0)$, and $\phi_m(-x)=\phi_m(x)$. It is also easy to see from (2.4) that

$$\frac{d}{dx}\phi_{m+1}(x) = 4(2m+1)x\phi_m(x). \tag{2.5}$$

We shall prove by induction that $\phi_m(x)$, and hence $\Omega_n^m(x)$ is nonnegative for $m=1, 2, \cdots$. We know from (1.3) that $\phi_1(x)\geqq 0$. Suppose $\phi_m(x)\geqq 0$. Then it follows from (2.5) and the fact that $\phi_{m+1}(0)=0$ that $\phi_{m+1}(x)$ is nonnegative if $x\geqq 0$. But $\phi_{m+1}(x)$ is even. Therefore $\phi_{m+1}(x)$ is everywhere nonnegative.

If we put $m=0$ in (2.5) we get the relation

$$\phi_1(x) = x^2\Delta_n(x) = 4\int_0^x xJ_n^2(x)dx;$$

the right-hand side is evaluated by means of [4, p. 151]

$$\int_0^x xJ_n^2(x)dx = 2\sum_{k=0}^{\infty}(n+1+2k)J_{n+1+2k}^2(x). \tag{2.6}$$

More generally, (2.5) yields the integral representation

$$\phi_m(x) = 2^m\frac{(2m)!}{m!}\int_0^x x\int_0^x x\int_0^x\cdots\int_0^x xJ_n^2(x)(dx)^m. \tag{2.7}$$

Now (2.6) and (2.7) yield

$$x^{2m}\Omega_n^{(m)}(x) = \frac{4^m(2m)!}{m!}\sum_{k=0}^{\infty}(n+m+2k)C_k(m)J_{n+m+2k}^2(x),$$

where $C_k(1)=1$ and

$$C_k(m) = \sum_{\substack{s_1+s_2+\cdots+s_m=k \\ s_1,s_2,\cdots s_m\geq 0}} \prod_{r=1}^{m-1}(n+r+2s_1+2s_2+\cdots+2s_r) \qquad (m \geqq 1).$$

The author is grateful to Professor L. Carlitz for pointing out that the coefficients $C_k(m)$ can be evaluated as

$$C_k(m) = \frac{1}{(m-1)!}(k+1)_{m-1}(n+k+1)_{m-1}. \tag{2.8}$$

The proof is as follows.

$$\begin{aligned} C_k(m+1) &= \sum_{s_1+s_2+\cdots+s_{m+1}=k} \prod_{r=1}^{m}(n+r+2s_1+2s_2+\cdots+2s_r) \\ &= \sum_{s=0}^{k}\sum_{s_1+s_2+\cdots+s_m=k-s}(n+m+2s_1+2s_2+\cdots+2s_m) \\ &\quad \cdot \prod^{m-1}(n+r+2s_1+\cdots+2s_r) \\ &= \sum_{s=0}^{k}(n+m+2s)C_s(m). \end{aligned}$$

We can now prove (2.8) by induction.

The justification for term by term integration to get (2.7) is by means of the inequality

$$|J_n(x)| \leqq (1/n!)\left|x/2\right|^n e^{|x|^2},$$

from which it follows that the series is uniformly convergent in every closed interval before and after each integration.

If we put $m=2$ in (1.2) we get

$$\begin{aligned} &x^4\{6J_n^2(x) - 8J_{n+1}(x)J_{n-1}(x) + 2J_{n+2}(x)J_{n-2}(x)\} \\ &\qquad = 2^6\cdot 3\sum_{k=0}^{\infty}(k+1)(n+k+1)(n+2+2k)J_{n+2+2k}^2(x). \end{aligned} \tag{2.10}$$

If in addition we put $n=0$ in (2.10) we obtain the result

$$x^4\{3J_0^2(x) + 4J_1^2(x) + 2J_2^2(x)\} = 2^6\cdot 3\sum_{k=1}^{\infty}k^3J_{2k}^2(x). \tag{2.11}$$

If in (1.2) we put $n=0$, we get

$$\binom{2m}{m}J_0^2(x) + 2\sum_{r=1}^{m}\binom{2m}{m-r}J_r^2(x) = \frac{4^m(2m)!}{x^{2m}m!(m-1)!}\sum_{k=0}^{\infty}(m+2k)\{(k+1)_{m-1}\}^2 J_{m+2k}^2(x). \tag{2.12}$$

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CLASSROOM NOTES

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ON THE SUM OF POWERS OF NATURAL NUMBERS

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The purpose of this paper is to find an expression for $\sum_{x=1}^{n} x^k$. This problem and its result are not new. Usually they are to be found in works on the calculus of finite differences.* However, it is the author's objective to present the solution from the point of view of elementary concepts of algebra in the hope of interesting those without training in the finite calculus technique.

Let us write the identity

$$(x-1)^{k+1} = x^{k+1} - \binom{k+1}{1}x^k + \binom{k+1}{2}x^{k-1} - \binom{k+1}{3}x^{k-2} + \cdots + (-1)^{k+1}, \tag{1}$$

and form the sum

$$\sum_{x=1}^{n}(x-1)^{k+1} = \sum_{x=1}^{n}x^{k+1} - \binom{k+1}{1}\sum_{x=1}^{n}x^k + \binom{k+1}{2}\sum_{x=1}^{n}x^{k-1} - \binom{k+1}{3}\sum_{x=1}^{n}x^{k-2} + \cdots + (-1)^{k+1}\sum_{x=1}^{n}1. \tag{2}$$

* See, for example, Charles Jordan, Calculus of Finite Differences, New York, 1957.

Rewrite the left member of (2) as

$$\sum_{x=1}^{n}(x-1)^{k+1}=\sum_{x=1}^{n}x^{k+1}-n^{k+1}. \tag{3}$$

Making use of (3) in (2) and solving for $\sum x^k$, one obtains

$$\begin{aligned}\sum x^k=\frac{n^{k+1}}{k+1}&+\frac{1}{2}\binom{k}{1}\sum x^{k-1}-\frac{1}{3}\binom{k}{2}\sum x^{k-2}\\&+\frac{1}{4}\binom{k}{3}\sum x^{k-3}+\cdots(-1)^m\frac{1}{m}\binom{k}{m-1}\sum x^{k-m+1}+\cdots.\end{aligned} \tag{4}$$

Equation (4) is a recursion relation and is taken as defining $\sum_{x=1}^{n}x^k$ so that $\sum x^{k-1}$ is obtained from (4) by replacing k by $k-1$. Similarly, $\sum x^{k-2}$ is obtained by replacing k by $k-2$, etc. Applying (4) to the second member of (4) one gets

$$\begin{aligned}\sum_{x=1}^{n}x^k=\frac{n^{k+1}}{k+1}&+\frac{1}{2}\binom{k}{1}\left[\frac{n^k}{k}+\frac{1}{2}\binom{k-1}{1}\sum x^{k-2}-\frac{1}{3}\binom{k-1}{2}\sum x^{k-3}+\cdots\right]\\&-\frac{1}{3}\binom{k}{2}\sum x^{k-2}+\frac{1}{4}\binom{k}{3}\sum x^{k-3}-\frac{1}{5}\binom{k}{4}\sum x^{k-4}+\cdots.\end{aligned} \tag{5}$$

Simplify (5) and obtain

$$\begin{aligned}\sum_{x=1}^{n}x^k=\frac{n^{k+1}}{k+1}&+\frac{n^k}{2}+\frac{k(k-1)}{12}\sum x^{k-2}-\frac{k(k-1)(k-2)}{24}\sum x^{k-3}\\&+\frac{k(k-1)(k-2)(k-3)}{80}\sum x^{k-4}+\cdots.\end{aligned} \tag{6}$$

Again apply (4) to third member of (6) and get

$$\sum_{x=1}^{n}x^k=\frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\frac{kn^{k-1}}{12}-\frac{k(k-1)(k-2)(k-3)}{720}\sum x^{k-4}+\cdots.$$

By repeated application of (4) one can conclude

$$\begin{aligned}\sum_{x=1}^{n}x^k=\frac{n^{k+1}}{k+1}&+\frac{n^k}{2}+\frac{kn^{k-1}}{12}+0\cdot n^{k-2}-\frac{1}{720}k(k-1)(k-2)n^{k-3}\\&+0\cdot n^{k-4}+\frac{k(k-1)(k-2)(k-3)(k-4)}{30{,}240}n^{k-5}+\cdots.\end{aligned}$$

This result may conveniently and symmetrically be written as

$$\sum_{x=1}^{n}x^k=\frac{n^{k+1}}{k+1}+\frac{n^k}{2}+\sum_{j=2}^{k}\frac{1}{j}B_j\binom{k}{j-1}n^{k-j+1},$$

where the B_j are the well-known Bernoulli numbers ($B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$,

$B_4=-\frac{1}{30}$, $B_6=\frac{1}{42}$, $\cdots$, and $B_{2k+1}=0$ for $k>0$).

Example. Find $\sum_{x=1}^{n} x^3$.

Solution.

$$\sum_{x=1}^{n} x^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{2}B_2\binom{3}{2}n^2 + \tfrac{1}{3}B_3\binom{3}{3}n$$

$$= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 = [\tfrac{1}{2}n(n+1)]^2.$$

NOTE ON EVALUATING CERTAIN REAL INTEGRALS BY CAUCHY'S RESIDUE THEOREM

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The evaluation of a real integral of the form $\int_0^\infty x^m dx/(x^n+a)$ where $a>0$, n is an even integer $\geqq 2$, and m is a nonnegative even integer $\leqq n-2$, by Cauchy's residue theorem is often presented in textbooks as follows. After observation that the value of the integral in question is half that of $\int_{-\infty}^{\infty} x^m dx/(x^n+a)$, the value of the latter is obtained by the usual limiting process, starting with an integral $\int_C z^m dz/(z^n+a)$, where z is complex and C is the closed contour running from $-R$ to R on the real axis, thence around the upper half of the circle $|z|=R$.

For all sufficiently large R the value of the integral around C is of course equal to $2\pi i$ times the sum of the residues at the zeros of z^n+a which lie in the upper half plane. An alternative procedure which avoids the addition of residues is given by Franklin.* This alternative, moreover, applies even when m and n are not integral. It does, however, require an extra change of variable.

The writers wish to present here a modification of the usual procedure which, while not as general in result as that given by Franklin in that m and n are still restricted to being nonnegative integers, does not require addition of residues and does not require an extra change of variable. Moreover, the integers m and n may now be odd or even and the integral from 0 to ∞ can be evaluated directly by the residue theorem without having recourse to the integral from $-\infty$ to ∞. The essential feature of the modification lies in using an appropriately chosen contour C which encloses *just one* of the zeros of z^n+a. This is done, for instance, in evaluating an integral of the form $\int_{-\infty}^{\infty} e^{pz}dx/(1+e^z)$.

Let C denote the contour made up of the segment of the real axis from $x=0$ to $x=R$, thence along $|z|=R$ to the ray $\arg z=\exp(2\pi i/n)$, thence along this ray to the origin. Thus for all sufficiently large R the contour C encloses just one of the zeros of z^n+a, where n is a positive integer, namely $z_1=a^{1/n}\exp(\pi i/n)$. Then we have

$$\int_C \frac{z^m dz}{z^n+a} = \frac{2\pi i z_1^m}{nz_1^{n-1}} = \frac{2\pi i}{na^{(n-m-1)/n}\exp[(n-m-1)\pi i/n]}.$$

* Methods of Advanced Calculus, New York, 1944, Ex. 83, p. 248.

When the integral around C is written, as usual, as the sum of three integrals, the integral along the ray $z=r\exp(2\pi i/n)$ may be written as

$$\int_R^0 \frac{r^m \exp[(m+1)2\pi i/n]dr}{r^n+a} = -\int_0^R \frac{x^m \exp[(m+1)2\pi i/n]dx}{x^n+a}.$$

Combining this last integral with the integral along the real axis from $x=0$ to $x=R$ and letting $R\to\infty$, we have, for $0<m+1<n$,

$$\int_0^\infty \frac{x^m dx}{x^n+a} = \frac{2\pi i}{\{1-\exp[(m+1)2\pi i/n]\}na^{(n-m-1)/n}\exp[(n-m-1)\pi i/n]}$$

$$= \frac{\pi}{na^{(n-m-1)/n}\sin[(m+1)\pi/n]}.$$

INTEGRATION

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Although we frequently think of indefinite integration as being more complex than the determination of roots of algebraic equations, we shall see that in some respects this is not the case. It is well known that solutions of some algebraic equations cannot be written in terms of radicals, and also that certain functions do not possess elementary integrals. However, due to the length and depth of the proofs of these facts, few texts go beyond merely exhibiting some examples. Confusing a sufficient condition with one which is both necessary and sufficient, many a student finds security in knowing that all equations of degree $\leqq 4$ have been solved. With respect to integration, however, he is well aware of his rather unhappy position, that of knowing that some functions cannot be integrated (in finite terms) but of possessing no way of determining whether a particular function is in this class or not. This latter situation need not be accepted, for Liouville obtained a test [1] which is both necessary and sufficient for the integrability of any one of a rather large class of functions. We shall recall this test and show both how it can be applied by the college freshman and its utility as an integration technique.

The foundation for the test is the following theorem which is sufficiently "natural" to be readily accepted and easily remembered.

LIOUVILLE'S THEOREM. *If $\int fe^g dx$ is an elementary function, f and g are rational functions of x, and the degree of $g>0$, then*

$$\int fe^g dx = Re^g \tag{1}$$

where R is a rational function of x.

This is actually a special case of Liouville's original theorem (see [1] page 114 or [2] page 47), but it is sufficiently general for our purposes.

In this note, by rational function we mean the quotient of two polynomials with coefficients in any field of characteristic zero, for example, the complex numbers. The term elementary function is more difficult to define (indeed, Ritt takes 12 pages to do so). However, the student is willing to accept as meaningful such statements as "the general algebraic equation of degree 5 cannot be solved in terms of radicals," although we know that "in terms of radicals" requires some preliminary discussion. Therefore, he finds no difficulty with the following definition: An elementary function is one which can be constructed by means of any finite combination of the operations addition, subtraction, multiplication, division, raising to powers, taking roots, forming trigonometric functions and their inverses, taking exponentials and logarithms. In short, no matter how complicated the function, if we can write down all of its terms, the function is elementary. (Actually, the construction of elementary functions includes the forming of algebraic functions, but it seems advisable to omit this generality for the beginning calculus student.)

We return to the test. If we wish to determine whether fe^g can be integrated (*i.e.* has an integral which is an elementary function, or, as we shall also say, $\int fe^g dx$ is elementary), we know, by Liouville's Theorem the form of the integral. Differentiating equation (1) and cancelling the nonzero e^g we find $f = R' + Rg'$ or, letting $R = P/Q$, where P and Q are relatively prime polynomials in x,

$$fQ^2 = P'Q - PQ' + PQg'. \tag{2}$$

Thus $\int fe^g dx$ is elementary if and only if there exist polynomials P and Q satisfying the differential equation (2).

Besides Liouville's theorem, the test requires only one further fact; namely, the following

LEMMA. *If the polynomial $f(x)$ has an r-fold zero at $x=\alpha$ and $r>0$, then $f'(x)$ has an $(r-1)$-fold zero at $x=\alpha$; in other words, if $f(x)=(x-\alpha)^r h(x)$ where $r>0$, $h(x)$ is a polynomial and $h(\alpha)\neq 0$, then $f'(x)=(x-\alpha)^{r-1}k(x)$ where $k(\alpha)\neq 0$.*

The proof of the lemma is a simple differentiation exercise which the student can supply.

It will make things easier if we define the term multiplicity. The number α is called a zero of the polynomial $f(x)$ of multiplicity r (or a root of $f(x)=0$ of multiplicity r) if $f(x)=(x-\alpha)^r h(x)$, where the polynomial $h(\alpha)\neq 0$. In terms of multiplicity the lemma reads: If α is a zero of the polynomial $f(x)$ of multiplicity $r>0$, then α is a zero of $f'(x)$ of multiplicity $r-1$.

By examining some of the examples most frequently quoted in texts, we shall show how easily the analysis of equation (2) can be carried out, even by the student who has not previously encountered differential equations.

Example 1: e^{-x^2}. If $\int e^{-x^2}dx$ is elementary, then $\int e^{-x^2}dx = Re^{-x^2}$ or $1 = R' - 2xR$. Letting $R = P/Q$, where P and Q are relatively prime polynomials and $Q\neq 0$, we find:

$$Q^2 = QP' - PQ' - 2xPQ \tag{1.2}$$

which is equation (2). Rearranging, we obtain

$$Q(Q - P' + 2xP) = - PQ'. \tag{1.3}$$

Let us assume that the degree of Q is positive. Then $Q=0$ has a root; let α be such a root and call its multiplicity $r(r>0)$. Since P and Q are relatively prime, $P(\alpha)\neq 0$. Now, α is a zero of the left side of (1.3) of multiplicity $\geqq r$ but α is a zero of the right side of multiplicity $r-1$. This is a contradiction, hence our assumption that the degree of Q is positive must be false. Q is a constant ($\neq 0$) which we can assume is unity.

From (1.3) we obtain

$$P' - 2xP = 1. \tag{1.4}$$

Since P is a polynomial in x, it is clear that the degree of $-2xP >$ degree of P', and the degree of $-2xP > 0$. The degree of the left side of (1.4) is always greater than the degree of the right side, which is a contradiction. We have proved that there is no polynomial P satisfying (1.4), hence no rational function satisfying (1.2). Consequently $\int e^{-x^2}dx$ is not elementary.

Example 2: e^{bx}/x, with b a nonzero constant. If $\int (e^{bx}/x)dx$ is elementary, then $\int (e^{bx}/x)dx = Re^{bx}$ or $(1/x) = R' + bR$. Letting $R = P/Q$, where P and Q are relatively prime polynomials, $Q\neq 0$, we find:

$$Q^2 = xQP' - xPQ' + xbPQ \tag{2.2}$$

$$Q(Q - xP' - bxP) = - xPQ'. \tag{2.3}$$

If we assume that Q has positive degree, then $Q=0$ has a root. Let α be such a root and call its multiplicity r. If $\alpha\neq 0$, we encounter the same contradiction met in the first example, that α is a zero of the left side of (2.3) of multiplicity $\geqq r$, while α is a zero of the right side of multiplicity $r-1$. Thus α must be zero, and $Q = cx^r$, for some $c\neq 0$. Putting this expression for Q in (2.2) we have $cx^{r+1}(cx^{r-1} - P' - bP) = -crx^rP$. Again there is a contradiction, for the number 0 is a zero of the left side of multiplicity $\geqq r+1$, while it is a zero of the right side of multiplicity r. Our assumption that Q has positive degree is no longer tenable; hence Q is a constant, which we can assume is unity.

From (2.3) we obtain

$$xP' + bxP = 1. \tag{2.4}$$

As before, since P is a polynomial in x, the degree of the left side = degree of $(bxP) > 0$ = degree of the right side. We have proved that there is no polynomial P satisfying (2.4), hence no rational function satisfying (2.2). Consequently $\int (e^{bx}/x)dx$ with $b\neq 0$ is not elementary.

Example 3: $(\sin x)/x$. It is clear that if $f(x) = u(x) + iv(x)$, where $u(x)$ and $v(x)$ are real valued functions, then

$$\Re \int f(x)dx = \int \Re f(x)dx = \int u(x)dx,$$

$$\Im \int f(x)dx = \int \Im f(x)dx = \int v(x)dx,$$

and if $\int f(x)dx$ is elementary, both $\int u(x)dx$ and $\int v(x)dx$ are elementary. ($\Re$ and $\Im$ stand for "the real and imaginary parts of," respectively.)

Although $(\sin x)/x$ is not in the form of Liouville's Theorem, by Euler's relation ($e^{ix}=\cos x+i \sin x$), we have $(\sin x)/x=\Im(e^{ix}/x)$. Since e^{ix}/x does not possess an elementary integral, by example 2, neither does $\Im(e^{ix}/x)=(\sin x)/x$.

Example 4: $1/\log x$. Again Liouville's theorem is not immediately applicable. If $y=\log x$, then $\int(1/\log x)dx=\int(e^y/y)dy$. By Example 2, the latter integral is not elementary, hence the same is true of the former.

Example 5: $(x^2+ax+b)e^x/(x-1)^2$, with a and b constants. Not only do the usual integration techniques require a considerable amount of skill, but there is no a priori assurance that we could find the integral, if it exists. Let us apply the test.

Assume $\int[(x^2+ax+b)e^x/(x-1)^2]dx=Re^x=Pe^x/Q$. Then

$$(x^2 + ax + b)Q^2 = (P'Q - Q'P + PQ)(x - 1)^2 \tag{5.2}$$

$$Q(Q(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P) = - Q'P(x - 1)^2. \tag{5.3}$$

Assume Q has positive degree, and let α be a zero of Q of multiplicity r. If $\alpha\neq 1$, α is a zero of the left side of multiplicity $\geqq r$, but a zero of the right side of multiplicity $r-1$. This is a contradiction, hence $\alpha=1$ and $Q=(x-1)^r$. Substituting this into (5.3) we find

$$(x - 1)^r[(x - 1)^r(x^2 + ax + b) - (x - 1)^2P' - (x - 1)^2P] = - r(x - 1)^{r+1}P.$$

Using the fact that the multiplicities of 1 as a zero of the left and right sides must be the same, we see that $r=1$; *i.e.* $Q=(x-1)$. In the last equation we can cancel a common factor of $(x-1)^2$ from both sides, giving

$$(x^2 + ax + b) - (x - 1)P' - (x - 1)P = - P,$$

$$(x - 1)P' + (x - 2)P = x^2 + ax + b.$$

P is clearly linear, $P=cx+d$, which means

$$cx - c + cx^2 + dx - 2cx - 2d = x^2 + ax + b.$$

Since these two polynomials are identical, the coefficients of like powers are the same, $c=1$, $d-c=a$, $-c-2d=b$, whence $b=-2a-3$.

Consequently $\int[(x^2+ax+b)e^x/(x-1)^2]dx$ is elementary if and only if $b=-2a-3$, in which case the integral is $e^x(x+a+1)/(x-1)+C$.

By using one unproved theorem, we have seen how it is possible for even the beginning calculus student to test the integrability of certain transcendental

functions, and the logical structure of the test, though not trivial, is sufficiently similar to others he has seen for it to be easily grasped (*e.g.*, the test for the rationality of say $\sqrt{2}$). Although it is obvious that a table of functions which cannot be integrated could be constructed by a careful analysis of examples, to do so for the student would be no better than is done at present. Rather we feel that the student, by applying the test to a few functions, will have a fruitful introduction to differential equations and will gain well-founded confidence in his ability to follow and reconstruct proofs requiring more than one or two steps. Also, the last example illustrates the usefulness of the test as an integration technique.

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THE INTEREST RATE IN INSTALLMENT CONTRACTS

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1. Introduction. In an installment contract* in which the time price differential is an add-on, the yield return to the lender increases as the contract gets longer up to a point. Thereafter the yield declines as the length of the contract increases.

Example. For a 6% add-on rate (*i.e.* $\frac{1}{2}$% of the original loan is added to the loan cost for each month of the duration of the loan) in an installment contract the effective rate is 8.98% per year for a 3-month contract and 10.21% for a 6-month contract. The yield builds up to a maximum of 11.13% around a 26-month contract. Thereafter it declines. The yield is 10.21% for a 120-month contract and it continues to decline to 6.2% in a 500-year contract.

The formula for the present value of a loan on a monthly basis is

$$B = Ra_{\overline{n}|} \text{ at } (i) \tag{1}$$

where B = cash price or present value of loan, R = monthly payment, n = number of months, and i = interest rate per month. Let c = per cent of add-on per month (*i.e.* at a 6% yearly add-on, $c = .005$); then $R = B(1+cn)/n$ and (1) becomes $B = B(1+cn)[1-(1+i)^{-n}]/ni$, or it can be put in the form

$$(1 + cn - in)(1 + i)^n - (1 + cn) = 0. \tag{2}$$

2. Maximum value of i for a constant value of c. The value of i as given by (2) is expressed approximately [1] [2] by

$$i_b = \frac{6cn}{3(n + 1) + cn(n - 1)}, \tag{3}$$

* This problem was presented to me by M. R. Neifeld, Beneficial Management Corporation, Morristown, New Jersey.

where

$$i - i_b < \frac{2(n-1)(n+2)}{9(n+1)^3} c^3 n^3.$$

From (3), we now obtain $\partial i_b/\partial n$ which is equated to zero. Solving for n gives

$$n_b = \sqrt{(3/c)}. \tag{4}$$

Since i_b is always smaller than i, and since the error, $i - i_b$, is an increasing function of n, it is apparent geometrically that n_b is too small. Although the value given by (4) is quite simple, it gives results which are about 1 too small. We present (4) only because it is useful in a more accurate determination of n (maximum).

A more exact value [2] of i is given by

$$i_c = \frac{2cn}{n+1}\left[\frac{3(n+1)+(n+2)cn}{3(n+1)+(2n+1)cn}\right] \tag{5}$$

where

$$i_c - i < \frac{2(n-1)(2n+1)(n+2)}{135(n+1)^4} c^4 n^4.$$

From (5), $\partial i_c/\partial n$ equated to zero gives

$$(c^2+3c)n^4 - 2c^2n^3 - (9+15c+2c^2)n^2 - 6(3+2c)n - 9 = 0. \tag{6}$$

Now (6) has only one positive root, n_c, for $c>0$. Since i_c is always greater than i, and since the error $i_c - i$ is an increasing function of n, it is apparent geometrically that n_c is always slightly large.

To find n_c, we take $n_a = \sqrt{(3/c)}+1$ as a starting point and apply Newton's formula, which gives

$$n = (\sqrt{(3/c)}+1) - \frac{18 + 24c\sqrt{(3/c)} + 30c + 6c^2\sqrt{(3/c)} + 3c^2}{18\sqrt{(3/c)} + 72 + 18c\sqrt{(3/c)} - 12c}$$

or

$$n_c = \sqrt{(3/c)} + 1 + \sqrt{(c/3)} + 2\sqrt{(c/3)^3} - \tfrac{1}{3}c^2 + \cdots.$$

Since in practical cases $0<i<.1$ and since $c<i$, we retain only the first three terms of n_c which gives

$$n_c \doteq \sqrt{(3/c)} + 1 + \sqrt{(c/3)} \text{ (maximum value)}. \tag{7}$$

3. Minimum value of c for a constant value of i. Solving (2) for c, gives

$$c = \frac{i}{1-(1+i)^{-n}} - \frac{1}{n},$$

$$\frac{\partial c}{\partial n} = \frac{1}{n^2} - \frac{\ln(1+i)}{s_{\overline{n}|} - a_{\overline{n}|}}. \tag{8}$$

Equating $\partial c/\partial n$ to zero, and expanding $\ln(1+i)$, $(1+i)^n$, and $(1+i)^{-n}$ to the seventh power of i, we solve the result for n^2 which gives

$$n^2 = \frac{1 - \frac{7}{6}i + \frac{7}{6}i^2 - \frac{127}{45}i^3 + \frac{7}{5}i^4}{\frac{1}{6}i - \frac{1}{3}i^2 + \frac{17}{36}i^3 - \frac{7}{12}i^4 + \frac{1}{180}n^2i^3 - \frac{1}{180}n^2i^4}. \tag{9}$$

Dividing the numerator of the right member of (9) by the denominator gives $n^2 = 6/i + \cdots$. Neglecting all but the first term gives $n^2 = 6/i$. Substituting this value for n^2 in the denominator of (9), we then divide again obtaining $n^2 = 6/i + 3.8 + \cdots$. If the first two terms are retained, and this value of n^2 is substituted in the denominator, there results by division

$$n \doteq \sqrt{(6/i + 3.8 - .32i)} \text{ (minimum value)}. \tag{10}$$

Formula (10) will be accurate to two decimal places as checked by central difference interpolation formula.

4. Some examples*

CRITICAL VALUE OF n	VALUE OF i	CORRESPONDING VALUE OF c
7.98	.10	.062444
17.43	.02	.011146
34.62	.005	.002644
84.87	1/1200	.0004264845

The critical value of n is a maximum for i with c constant or a minimum for c with i constant.

5. Discount contract. In cases of a discount loan, the sum of the payments is B and the present value of the loan is $B - Bcn$. For example, if $\frac{1}{2}\%$ a month is charged, for a loan of 12 months \$94 would be received by the borrower for the total sum of 12 payments of \$100. In general, the formula for this type of loan is

$$B(1 - cn) = (B/n)a_{\overline{n}|} \tag{11}$$

or $a_{\overline{n}|} = n - cn^2$ for $0 < cn < 1$, $0 < a_{\overline{n}|} < n$. As n increases, with c constant, $a_{\overline{n}|}$ decreases monotonically relative to n, and i increases correspondingly, hence there is no relative maximum in this case.

* By use of the recursion formula

$$c_1 = 1,\ c_{n+1} = \frac{[c_n + (1/n)](1+i)}{c_n + (1/n) + 1} - \frac{1}{n+1},$$

which is obtained by combining the formulas $c_n = (1/a_{\overline{n}|}) - (1/n)$, $c_{n+1} = (1/a_{\overline{n+1}|}) - (1/(n+1))$ and making use of the relation $a_{\overline{n}|} + 1 = (1+i)a_{\overline{n+1}|}$. Professor Gerald Weeg, Michigan State University, used the MYSTIC computer to make a table of add-on equivalent of effective periodic interest rates to 8 decimal places for i from $\frac{1}{4}\%$ to 2% by $\frac{1}{4}\%$ and n from 1 to 102.

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A VERY INDEPENDENT AXIOM SYSTEM

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1. Introduction. One of the most popular mathematics courses at the University of Michigan is Mathematics 195, *Foundations of Mathematics*. This course has been given for over two decades by Professor R. L. Wilder and his excellent book [7], is used as the text. I wish to thank the students in my Mathematics 195 course this semester (Spring 1960) for their assistance in our stumbling onto an apparently new and stronger concept of the independence of an axiom system.

Let $S=\{A_1, \cdots, A_n\}$ be a system of axioms over a given collection of primitives. We shall assume throughout that this axiom system is *satisfiable*, that is, that it has a model. Following Wilder [7], an axiom A in S is *independent* if the modified system $(S-A)+\sim A$ is also satisfiable; and S itself is *independent* if each of its axioms is independent. Continuing, the axiom system S is *completely independent* if for any subset S_1 of axioms, the modified axiom system $(S-S_1)+\sim S_1$ has a model.

There was considerable interest in constructing completely independent axiom systems for groups, boolean algebras, and other algebraic systems in the beginning decades of this century; see for example the works of Bernstein [1] and Huntington [4].

In the above definition of an independent axiom A it is only necessary to have a model in which A does *not always* hold. However, it is also sometimes possible to construct a model in which all the other axioms hold and A *never* holds. In this case, we shall say that A is a *very independent axiom*. An axiom *system* S is *very independent* if each of its axioms is very independent. We restrict consideration to axioms of the form "p implies q": we admit the possibility of taking p as the universally true statement. The hypothesis p of this statement may be a condition of the form: x and y are elements of some set U. We shall say that an axiom A of the form "p implies q" never holds in a model M if the hypothesis p occurs at least once in M and furthermore whenever p is true, q is false, *i.e.*, p implies not q.

In an entirely analogous manner, we say that the axiom system S is *absolutely independent* if for every subset S_1 of axioms, there is a model in which $S-S_1$ holds and each axiom in S_1 never holds.

A *relation* is a set of ordered couples. For any relation R, we write aRb to indicate that $(a, b)\in R$. The elements $a, b, c, \cdots$ which may be in the relation

R to each other constitute the *field* F of the relation. With this notation, one usually has the following definitions which hold for all $x, y, z \in F$:

R is *reflexive* if xRx;
R is *symmetric* if xRy implies yRx;
R is *transitive* if xRy and yRz imply xRz.

The above definition of transitivity is given in the books [2, 3, 7]. However, there is an alternative definition of a transitive relation in König [5], and Lewis and Langford [6]:

R is *distinctly transitive* if for any three distinct elements $x, y, z \in F$, xRy and yRz imply xRz.

We now include the corresponding properties of relations which are opposite to conditions r, s, t (reflexive, symmetric, distinctly transitive).

R is *irreflexive* if xRx does not hold for any $x \in F$.
R is *asymmetric* if whenever $x \neq y$ and xRy holds, yRx does not;
R is *intransitive* if for distinct $x, y, z \in F$, xRy and yRz imply that xRz is false.

We abbreviate the last three properties by the symbols $\bar{r}$, $\bar{s}$, $\bar{t}$ respectively. An *anti-equivalence relation* is irreflexive, asymmetric, and intransitive ($\bar{r}$, $\bar{s}$, $\bar{t}$). We do not know of any general theorems for such relations.

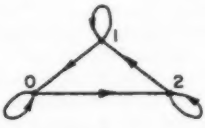
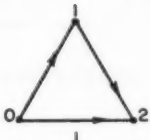
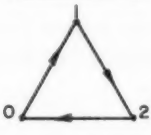
2. A very independent axiom system. An *equivalence relation* is usually defined as one which is reflexive, symmetric, and transitive. In this form, the statement is often made that symmetry and transitivity imply reflexivity. This assertion is usually accompanied by considerable discussion since it needs to be qualified by appropriate conditions in order to be incontrovertibly true; but this does not concern us here. We now state an axiom system for equivalence relations which is not only independent, but is very independent; and indeed, even absolutely independent!

The primitives or undefined terms in this system are a nonempty set F of elements $x, y, z, \cdots$ and a relation R whose field is F. The axioms are:

r. R is reflexive, s. R is symmetric, t. R is distinctly transitive.

In order to show that R is very independent, it is necessary to construct four models, one for each of the sets of conditions rst, $\bar{r}st$, $r\bar{s}t$, $rs\bar{t}$. Then in order to verify the further statement that this axiom system is absolutely independent, it is necessary to produce four additional models which satisfy: $\bar{r}\bar{s}t$, $\bar{r}s\bar{t}$, $r\bar{s}\bar{t}$, $\bar{r}\bar{s}\bar{t}$.

We shall now exhibit eight relations whose field is $\{0, 1, 2\}$ which satisfy the conditions listed above. It is convenient to include a graphical representation of these relations in which there is a directed line from point i to point j if and only if iRj and there is a directed line from point i to itself if and only if iRi.

Conditions	xRy where $F=\{0, 1, 2\}$	Diagram of the relation on $\{0, 1, 2\}$
rst	$x=y$ or $x\neq y$	
$\bar{r}st$	$x\neq y$	
$r\bar{s}t$	$x\geqq y$	
$rs\bar{t}$	$\lvert y-x\rvert \leqq 1$	
$r\bar{s}\bar{t}$	$y\neq(x+1) \pmod 3$	
$\bar{r}s\bar{t}$	$\lvert y-x\rvert =1$	
$\bar{r}\bar{s}t$	$x<y$	
$\bar{r}\bar{s}\bar{t}$	$y=(x+1) \pmod 3$	

It is interesting to write eight corresponding relations on the field of all people; some of these are given in Copi [3]. For example, "sibling" is $\bar{r}st$, "uncle" is $\bar{r}\bar{s}\bar{t}$, "taller than" is $\bar{r}\bar{s}t$, and "has the same first name" is rst. Continuing, "at least as tall" is $r\bar{s}t$, "within one year of age" is $rs\bar{t}$, "exactly one year difference" is $\bar{r}s\bar{t}$, and "same age or exactly one year older" is $r\bar{s}\bar{t}$.

3. Problem. What other very independent axiom systems are there? It may be regarded as fortuitous that a very independent axiom system was found at all. We note that the usual axiom system for groups is independent but not very. For taking the primitives as a set with a binary operation, and the postulates as closure, associativity, the existence of an identity element, and the existence of inverse elements, we see that the associative law certainly holds whenever at least one of the three elements involved is the identity. Hence, whenever the identity postulate does hold, associativity cannot always fail.

Recently, Evans [8] gave a postulational description of a number theory in which a collection of numbers generated by a set corresponding to the natural numbers is closed under addition. These have the property that the associative law for addition never holds, and neither does the commutative law.

Euclid's parallel postulate is very independent! This is shown by both kinds of non-Euclidean geometry. The postulate states that if L is any line and A is a point not on L, then there exists a unique line M containing A and disjoint from (parallel to) L. In the Bolyai-Lobachewsky geometry, there is always more than one such line M, in fact an infinite number. But in Riemannian geometry, there is never any such line M, for any two lines meet in exactly one point.

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LOGARITHMIC NUMBERS AND SOME THEOREMS ON PERMUTATIONS

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The author has defined the *related logarithmic polynomials** $A_r^{(n)}(t)$ by the generating function

$$\sum_{r=0}^{\infty} A_r^{(n)}(t)x^r/r! = (x^{n-1}e^{-xt})/(1-x^n). \tag{1}$$

Whence expanding and equating the coefficients of x^r we get

* See [1], where the author has defined the logarithmic polynomials $G_r^{(n)}(t)$ by $e^{-xt}\log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(t)x^r/r!$. If this expression is differentiated with respect to x, (1) follows on putting $A^{(n)}(t) = (tG^{(n)}(t)+G_{r+1}^{(n)}(t))/n = A_r^{(n)}(t)$.

$$A_r^{(n)}(t)/r! = \sum_i (-t)^{r-in+1}/(r-in+1)!. \tag{2}$$

The special cases $t=1$ and $t=-1$ of $A_r^{(n)}(t)$ are called *logarithmic numbers.*

We first show that canonical generators for $A_r^{(n)}(-1)$ and $A_r^{(n)}(1)$ are

$$A_r^{(n)}(-1) = r^{(n)} A_{r-n}^{(n)}(-1) + r^{(n-1)}, \tag{3}$$

$$A_r^{(n)}(1) = r^{(n)} A_{r-n}^{(n)}(1) + (-1)^{r-n+1} r^{(n-1)}, \tag{4}$$

where $r^{(n)} = r(r-1)\cdots(r-n+1)$. Putting $t=-1$ in (2) we get

$$\begin{aligned} A_r^{(n)}(-1) &= r! \sum_i 1/(r-in+1)! = r^{(n)}(r-n)! \sum_i 1/(r-in+1)! \\ &= r^{(n)}(r-n)! \sum_i 1/(r-n-in+1)! + r^{(n)} \cdot (r-n)!/(r-n+1)! \\ &= r^{(n)} A_{r-n}^{(n)}(-1) + r^{(n-1)}. \end{aligned}$$

This proves (3) and, similarly, (4) can be proved.

The case $n=1$ is of special interest in the present paper. Putting $n=1$, $t=-1$ and $t=1$ in (2), we get after some simplification

$$A_r(-1) = \sum_{t=0}^{r} {}_rP_t, \qquad A_r(1) = \sum_{t=0}^{r} (-1)^t {}_rP_t,$$

where ${}_rP_t \equiv r!/(r-t)!$ Putting $n=1$ in (3) and (4) we get

$$A_r(-1) = rA_{r-1}(-1) + 1, \qquad A_r(1) = rA_{r-1}(1) + (-1)^r.$$

Now referring to Mullin's paper [2], we note that his $\Phi(r)$ is the same as $A_r(-1)$ so that the logarithmic numbers include $\Phi(r)$ as a special case. It was proved in [2] that

$$\Phi(r) \sim r!e \quad \text{or} \quad A_r(-1) \sim r!/e. \tag{5}$$

In this paper we prove that

$$A_r(1) \sim r!/e. \tag{6}$$

We have

$$\begin{aligned} A_r(1) &= r!\left\{\sum_{i=1}^{r+1} (-1)^{r-i+1}/(r-i+1)!\right\} = r! \sum_{t=0}^{r} (-1)^t/t! \\ &= r!\left\{\sum_{t=0}^{\infty} (-1)^t/t! - \sum_{t=r+1}^{\infty} (-1)^t/t!\right\} = r!\{1/e + o(1)\}, \end{aligned}$$

which proves (6).

We get a simple asymptotic formula for the A's by dividing (5) by (6):

(7) $$A_r(-1)/A_r(1) \sim e^2.$$

Using Stirling's formula for $r!$, (5) and (6) may be written

(8) $$A_r(-1) \sim r^{r+\frac{1}{2}}e^{-r+1}\sqrt{(2\pi)},$$

(9) $$A_r(1) \sim r^{r+\frac{1}{2}}e^{-r-1}\sqrt{(2\pi)}.$$

Multiplying (8) and (9) we get, using (7),

$$A_r(-1)A_r(1) \sim r^{2r+1}e^{-2r}(2\pi) \sim 2\pi r^{2r+1}\{A_r(1)/(A_r(-1)\}^r.$$

Hence $\frac{1}{2}r^{-2r-1}\{A_r(-1)\}^{r+1}/\{A_r(1)\}^{r-1} \sim \pi$.

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A NOTE ON HAUSDORFF SEPARATION

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The examples usually given as instances of topological spaces that have T_1-separation but not T_2-separation (Hausdorff) also have the property that some compact subset is not closed. This with the classic result concerning closedness of compact subsets of a Hausdorff space suggests the question of the equivalence of Hausdorff separation and the condition that the class of compact subsets be a subclass of the class of the closed subsets of a given space. The following is a simple result of this type and may be of some use in an introductory course in point set topology.

THEOREM. *If X is a space satisfying the first axiom of countability, then a necessary and sufficient condition that X be a Hausdorff space is that the class of compact subsets of X be a subclass of the class of closed subsets of X.*

Only the sufficiency need be considered here.

Since points are compact, it is immediate that X must be at least T_1. Also it can be assumed that the neighborhood base $\{V_n/n=1, 2, \cdots\}$ at each point is such that $V_n \subseteq V_m$, $n \geqq m$. Suppose there exist points x and y such that there are no disjoint pairs of neighborhoods of x and y respectively. Then a sequence $\{x_n/n=1, 2, \cdots\}$ may be selected by choosing each x_n in the intersection of the nth sets of the neighborhood bases of x and y. The set $\{x_n/n=1, 2, \cdots\} \cup [x]$ is compact but is not closed since y is an accumulation point.

That the assumption of a local countable base or some other restriction is necessary is seen from the following example.

Let X be an uncountable set with a topology such that a set is open if and only if it is X, the null set or the complement of a countable set. The space is not Hausdorff and does not satisfy the first axiom of countability. However the only compact sets are finite sets and hence closed.

THE BETA-GAMMA FUNCTION IDENTITY

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The Beta-Gamma function identity

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}$$

is traditionally demonstrated by use of a double integral in polar coordinates. There is a more compact proof which uses the Laplace transformation.

In general,

$$L\{t^{k-1}\} = \Gamma(k)s^{-k} \qquad (k > 0). \tag{1}$$

Using the convolution integral and the substitution $x = t^{-1}\tau$,

$$\begin{aligned}\Gamma(p)s^{-p}\Gamma(q)s^{-q} &= L\left\{\int_0^t \tau^{p-1}(t - \tau)^{q-1}d\tau\right\} \\ &= L\left\{t^{p+q-1}\int_0^1 x^{p-1}(1 - x)^{q-1}dx\right\} = L\{t^{p+q-1}B(p, q)\}.\end{aligned} \tag{2}$$

But

$$\Gamma(p + q)s^{-p-q}B(p, q) = L\{t^{p+q-1}B(p, q)\}. \tag{3}$$

From equations (2) and (3),

$$\Gamma(p)\Gamma(q) = \Gamma(p + q)B(p, q); \qquad \text{Re}\,(p),\ \text{Re}\,(q) > 0. \tag{4}$$

This is the desired identity.

COMPLETE CONTINUITY FOR FUNCTIONS

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Introduction. Levine* has defined strong continuity for functions. A function $f: (X, S) \to (Y, T)$, where S and T are topologies for X and Y respectively, is strongly continuous if and only if the inverse image of every set in the range is open in the domain. It is to be recalled that continuity demands only that the inverse images of open sets in the range-space be open in the domain-space. It is the purpose of this note first to generalize slightly Levine's Theorem 2 and then to redescribe strong continuity in terms of different topologies for Y.

1. Quasi-constant functions.

DEFINITION 1. *A function f defined on a space (X, S) is called quasi-constant if and only if f is constant on each quasi-component of (X, S).*

THEOREM 1. *Let $f: (X, S) \to (Y, T)$ be strongly continuous, then f is quasi-constant.*

* Norman Levine, Strong continuity in topological spaces, this MONTHLY, vol. 67, 1960, p. 269.

Proof. Let a and b be two points of X that lie in the same quasi-component of X. Assume that $f(a)=\alpha\neq\beta=f(b)$. Now $f^{-1}(\alpha)\cap f^{-1}(\sim\{\alpha\})=\varnothing$ where $\sim\{\alpha\}$ denotes the complement of α in Y. $X=f^{-1}(\alpha)\cup f^{-1}(\sim\{\alpha\})$ and $f^{-1}(\alpha)$ and $f^{-1}(\sim\{\alpha\})$ are both open. Thus $f^{-1}(\alpha)\cup f^{-1}(\sim\{\alpha\})$ is a separation of X. Further, a is in $f^{-1}(\alpha)$ and b is in $f^{-1}(\sim\{\alpha\})$; this contradicts the fact that a and b lie in the same quasi-component of X. Hence f is quasi-constant.

COROLLARY 1.1. *If $f\colon (X, S)\to(Y, T)$ is strongly continuous then f is constant on the components of X.*

Proof. Components are contained in quasi-components.
Corollary 1.1 is a rephrasing of Levine's Theorem 2.

COROLLARY 1.2. *If (X, S) is a connected space then the only strongly continuous functions are the constant functions.*

Proof. (X, S) consists of one component, X.

Example. Let (X, S) be the subspace of the space, E^2, (the euclidean plane) based on the set, $\{(x, y)|x=1/n$ and $0\leqq y\leqq 1$ for $n=1, 2, \cdots\}\cup\{(0, 0)\}\cup\{(0, 1)\}$ (Fig. 1). Now, $\{(0, 1)\}$ constitutes a component of (X, S) as also

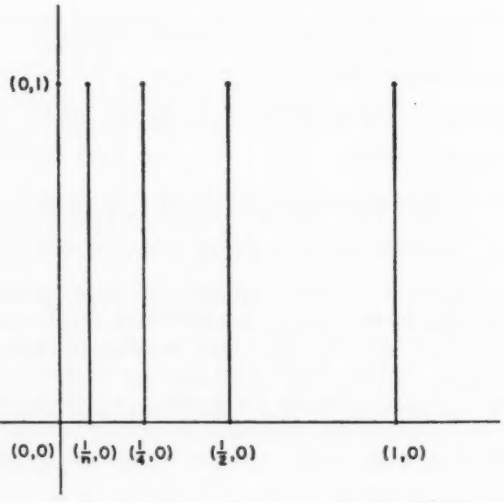

FIG. 1

does $\{(0, 0)\}$. However, $\{(0, 1)\}\cup\{(0, 0)\}$ constitutes one quasi-component, since in any separation $A\cup B$ of X, (0, 1) and (0, 0) would both have to be in the same part of this separation. Hence, if f is to be strongly continuous on (X, S) not only does each of the verticals, $\{(x, y)|x=1/n$ and $0\leqq y\leqq 1\}$ have

to have a one point image but (0, 1) and (0, 0) must be assigned the same image. Of course, not every quasi-constant function is strongly continuous. Levine's example can be used to show this since in his example components also happen to be quasi-components or in the above space define $f(1/n, y) = (1/n, 0)$ and $f(0, 0) = f(0, 1) = (0, 1)$. Assign the relative topology to the range of f. f is quasi-constant but, since $f^{-1}(\{(0, 1)\})$ is not open in the domain (X, S), f is neither strongly continuous nor even continuous.

2. Completely continuous functions. In introductory analysis, continuity is usually considered as an intrinsic property of a function; this is because the topologies on the domains and ranges are fixed. However, any function $f: X \to Y$ can be made continuous by assigning "right" topologies S and T to X and Y, respectively. Now, continuity for f depends on the relative "size" of S to T and T to S, since if f is to be continuous the inverse image of every set in T must be in S. Thus so long as S is "big enough" relative to T or equivalently T is "small enough" relative to S, f is continuous; otherwise f, the same f, is not continuous. For example, if $X = Y =$ set of all real numbers, if S consists of X and $\varnothing$ only, *i.e.*, S is the trivial topology for X, and if T consists of all subsets of Y, *i.e.*, T is the discrete topology for Y, and if f is the "identity" map $i: (X, S) \to (Y, T)$, where $i(x) = x$, then i is not continuous [*e.g.*, $\{0\}$ is open in (Y, T) but $i^{-1}(\{0\})$ is not open in (X, S)]. Now if we assign the topology T to X and the topology S to Y, then the "same" function $i: X \to Y$, where $i(x) = x$, is continuous.

THEOREM 2. *If f is a mapping from a topological space (X, S) into a topological space (Y, T) where S is the discrete topology for X, then f is strongly continuous.*

Proof. Since every subset of X is open in (X, S), every inverse image set is open in (X, S) and f is strongly continuous.

THEOREM 3. *A one-to-one mapping from a space (X, S) into a space (Y, T) is strongly continuous if and only if S is the discrete topology for X.*

Proof. The "if" part follows from Theorem 2. Conversely if f is strongly continuous, every one-point subset $\{x\}$, the inverse image of $f(x)$, is open in (X, S). Hence, every subset of X is open in (X, S).

COROLLARY 3.1. *A homeomorphism, f, from a space (X, S) onto a space (Y, T) is strongly continuous if and only if S and T are the discrete topologies for X and Y, respectively.*

Proof. If f is strongly continuous, by Theorem 3, every subset in X is open. Since f is a homeomorphism, open sets have open images, hence every subset of Y is open.

DEFINITION 2. *A mapping f defined on a space (X, S) is called completely continuous on (X, S) if and only if f is continuous with each and every topology on the range.*

THEOREM 4. *A function f from a space (X, S) into a space (Y, T) is strongly continuous if and only if f is completely continuous on (X, S).*

Proof. If f is strongly continuous, then since the inverse image of every subset of Y is open in X, f is continuous, using the discrete topology on Y. So f is completely continuous. Conversely, if f is completely continuous on (X, S), then f is continuous with any topology on the range, including, then, the discrete topology. Hence, it follows that the inverse image of every subset of Y is open in (X, S) and so f is strongly continuous.

Thus, for strong continuity, the discrete topology on the domain is sufficient, quasi-constancy is necessary and complete continuity is necessary and sufficient.

RECURSION FORMULAS FOR DERIVATIVES OF TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Recursion formulas for the values at the origin of nth-derivatives of trigonometric and hyperbolic functions are of interest in connection with Maclaurin Series expansions. In this paper we establish two such formulas, each applicable with only a single change to many functions, and more economical than those generally known.

We wish to evaluate $f^{(n)}(0) = A_n$ for certain functions $f(x)$. To accomplish this we consider the product $\phi(x) = M(x)f(x)$ where $M(x) = \cos x \cosh x = \Re[\cosh(1+i)x]$. Setting $M^{(k)}(0) = M_k$, we see that $M_{2r+1} = 0$, $r = 0, 1, 2, \cdots$, and $M_{2r} = \Re[(1+i)^{2r} \cosh(1+i)x]_0 = \Re[(1+i)^{2r}] = (-4)^t$ when $r = 2t$, 0 when $r = 2t+1$, $t = 0, 1, 2, \cdots$.

Applying Leibniz's formula for the nth derivative of a product to $\phi(x) = M(x)f(x)$, assuming all A_k to exist, and setting $\phi^{(n)}(0) = B_n$, we obtain

$$B_n = \sum_{t=0}^{[\frac{1}{4}n]} \binom{n}{4t} (-4)^t A_{n-4t},$$

which may be written in the form

$$\text{(I)} \qquad A_n = B_n + 4\binom{n}{4}A_{n-4} - 4^2\binom{n}{8}A_{n-8} + 4^3\binom{n}{12}A_{n-12} - \cdots$$

This is a generic recursion formula applicable to functions $f(x)$ for which the corresponding B_n is known. For example, if $f(x) = \sec x$, $\phi(x) = \cos x \cosh x \sec x = \cosh x$, and then $B_{2t} = 1$, $B_{2t+1} = 0$, $t = 0, 1, 2, \cdots$. Then (I) gives $A_{2t+1} = 0$ and

$$A_{2t} = 1 + 4\binom{2t}{4}A_{2t-4} - 4^2\binom{2t}{8}A_{2t-8} + 4^3\binom{2t}{12}A_{2t-12} - \cdots,$$

whence $A_0=1$, $A_2=1$, $A_4=5$, $A_6=61$, $\cdots$. These are the numerical values of the Euler numbers. The recursion formula is very effective. To obtain A_{10}, for example, it is merely necessary to know A_2 and A_6, since $A_{10}=1+840A_6-720\,A_2=50{,}521$.

If $f(x)=\operatorname{sech} x$, $\phi(x)=\cos x$, whereupon $B_n=0$ for $n=2t+1$, 1 for $n=4t$, and -1 for $n=4t+2$. Consequently the recursion formula for $A_n'=[D^n \operatorname{sech} x]_0$ is exactly that for $A_n=[D^n \sec x]_0$ when $n=2t+1$ or $4t$, wherefore $A'_{2t+1}=A_{2t+1}=0$, and $A'_{4t}=A_{4t}$. When $n=4t+2$, $B_n'=-1=-B_n$. Addition of (I) for $\sec x$ and $\operatorname{sech} x$ then shows that $A_{4t+2}+A'_{4t+2}=0$, or $A'_{4t+2}=-A_{4t+2}$. Hence the derivatives of $\operatorname{sech} x$ at $x=0$ are those of $\sec x$, but with alternating signs.

To obtain formulas for the derivatives at $x=0$ of $\sec x \operatorname{sech} x$, $\tan x$, $\tanh x$, $e^x \sec x$, and $e^x \operatorname{sech} x$, it is merely necessary to substitute in (I) the corresponding values of B_n. These can be obtained by the method used above in finding M_k. They are given in Table I.

TABLE I

$f(x)$	$\phi(x)=\cos x \cosh x\, f(x)$	B_n
$\sec x \operatorname{sech} x$	1	$1,\ n=0$ $0,\ n>0$
$\tan x$	$\sin x \cosh x$	$0,\ n=2t$ $(-4)^t,\ n=4t+1$ $2(-4)^t,\ n=4t+3$
$\tanh x$	$\cos x \sinh x$	$0,\ n=2t$ $(-4)^t,\ n=4t+1$ $-2(-4)^t,\ n=4t+3$
$e^x \sec x$	$e^x \cosh x$	$1,\ n=0$ $2^{n-1},\ n>0$
$e^x \operatorname{sech} x$	$e^x \cos x$	$(-4)^t,\ n=4t,\ 4t+1$ $0,\ n=4t+2$ $-2(-4)^t,\ n=4t+3$

A companion formula to (I), obtained by the same method but using $M(x)=\sin x \sinh x$, is

$$\text{(II)} \qquad \binom{n}{2} A_{n-2} = \tfrac{1}{2}B_n + 4\binom{n}{6} A_{n-6} - 4^2\binom{n}{10} A_{n-10} + \cdots.$$

Use has been made of the fact that $\sin x \sinh x = \Im[\cosh(1+i)x]$ which gives $M_{4t+2}=(-1)^t 2^{2t+1}$, $M_n=0$, $n\neq 4t+2$. This new recursion formula is useful in determining the coefficients in the expansion of such functions as $x \csc x$, $x \operatorname{csch} x$, $x^2 \csc x \operatorname{csch} x$, $x \cot x$, and $x \coth x$. The corresponding values of B_n are given in Table II.

TABLE II

$f(x)$	$\phi(x)=\sin x \sinh x f(x)$	B_n $(B_0=0)$
$x \csc x$	$x \sinh x$	$0,\ n=2t+1$ $n,\ n=2t+2$
$x \operatorname{csch} x$	$x \sin x$	$0,\ n=2t+1$ $n,\ n=4t+2$ $-n,\ n=4t+4$
$x^2 \csc x \operatorname{csch} x$	x^2	$2,\ n=2$ $0,\ n\neq 2$
$x \cot x$	$x \cos x \sinh x$	$0,\ n=2t+1$ $n(-4)^t,\ n=4t+2$ $-2n(-4)^t,\ n=4t+4$
$x \coth x$	$x \sin x \cosh x$	$0,\ n=2t+1$ $n(-4)^t,\ n=4t+2$ $2n(-4)^t,\ n=4t+4$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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REPORT ON THE PROGRAM OF VISITING LECTURERS TO COLLEGES, 1960–61

R. A. ROSENBAUM, Wesleyan University

This note is a supplement to one by Rothwell Stephens (this MONTHLY, vol. 67, 1960, pp. 463–465).

During 1959–60, the six visiting lecturers spent a total of 257 days at 129 institutions, the average length of a visit thus being two days. In addition, there were 35 visiting mathematicians who lectured for a total of 83 days, in most instances making one-day visits.

The itineraries of the visiting lecturers in the current academic year are as follows (minor changes may still be made):

R. D. Anderson

Oct. 31, Nov. 1	Grinnell College
Nov. 2	Iowa State Teachers College
Nov. 3, 4	Wartburg College
Nov. 7, 8	Luther College
Nov. 9	College of St. Teresa, Minnesota
Nov. 10	St. Mary's College, Minnesota
Nov. 11	Hamline University
Nov. 15	St. Cloud College
Nov. 16	College of St. Benedict
Nov. 17	St. John's University, Minnesota
Nov. 28, 29	East Texas State College
Nov. 30, Dec. 1	North Texas State College
Dec. 2	East Central State College
Dec. 5, 6	Oklahoma City University
Dec. 7	Central State College, Oklahoma
Dec. 8, 9, 10	University of Oklahoma
Dec. 12	Kansas State University
Dec. 13, 14	Washburn University
Dec. 15	Kansas University
Dec. 16	Baker University

Charles W. Curtis

Feb. 2, 3	Earlham College
Feb. 6, 7, 8	Mississippi State University
Feb. 9, 10	Millsaps College
Feb. 13, 14	Mississippi Southern College
Feb. 15, 16	Loyola University, Louisiana
Feb. 17, 18	Louisiana-Mississippi Section of M.A.A.
Feb. 20, 21	Talladega College
Feb. 23, 24	University of the South
Mar. 4, 6	Mundelein College
Mar. 7, 8	Rosary College
Mar. 9, 10	Saint Xavier College
Mar. 13, 14	Bradley University
Mar. 16, 17	Southwestern at Memphis
Mar. 20, 21	Anderson College
Mar. 23, 24	Marquette University
Mar. 27, 28	Wisconsin State College, River Falls
Mar. 29, 30	Wisconsin State College, Eau Claire

Andrew M. Gleason

Apr. 3, 4	Vanderbilt University
Apr. 6, 7	Berea College
Apr. 10, 11	Indiana University
Apr. 13, 14	Eastern Illinois University
Apr. 17, 18	Purdue University
Apr. 19	Butler University
Apr. 20, 21	Indiana Technical College
Apr. 24, 25, 26	University of Wisconsin
Apr. 28	Western Illinois University

H. H. Goldstine

Oct. 31, Nov. 1, 2	University of Buffalo
Nov. 3, 4	St. Bonaventure University
Nov. 7, 8, 9	University of Ottawa, Carleton University
Nov. 28, 29, 30	Wesleyan University, Trinity College, Connecticut
Dec. 1, 2, 3, 4, 5	Smith College, University of Massachusetts, Mt. Holyoke College
Dec. 7, 8, 9	Dartmouth College
Apr. 3, 4, 5	Rutgers, The State University
Apr. 6, 7	Haverford College
Apr. 10, 11	Franklin and Marshall College
Apr. 12, 13	Bucknell University
Apr. 14	Geneva College
Apr. 24, 25	Merrimack College
Apr. 26, 27, 28	College of the Holy Cross, Clark University, Worcester Polytechnic Institute
May 1, 2	Southern Illinois University
May 4, 5	Auburn University, Alabama

Leon Henkin

Apr. 5, 6, 7	Miami University, Ohio
Apr. 10, 11	Denison University
Apr. 13, 14	Ohio Wesleyan University
Apr. 17, 18, 19, 20	American University, Catholic University, George Washington University, Howard University, District of Columbia Teachers College
Apr. 24, 25	Oberlin College
Apr. 27, 28	University of Dayton
May 1	University of Michigan
May 2, 3	Hope College
May 4, 5	Central Michigan College

Leo Moser

Mar. 16, 17	Eastern New Mexico University
Mar. 20, 21	Pomona College
Mar. 22, 23, 24	University of California, Santa Barbara
Mar. 27, 28	California State Polytechnic College
Apr. 6, 7	George Pepperdine College
Apr. 10, 11	University of Nevada
Apr. 12, 13, 14	University of Arizona
Apr. 17, 18	Colorado State University
Apr. 20, 21	Utah State University
Apr. 24, 25	Humboldt State College
May 1, 2	San Jose State College, University of Santa Clara
May 3	St. Mary's College of California
May 4, 5	University of California, Davis
May 8, 9	Sacramento State College
May 11, 12	Willamette University
May 15, 16	Montana State University
May 17	Northwest Nazarene College
May 18, 19	Montana State College
May 22, 23	Washington State University
May 25, 26	University of Idaho

Hartley Rogers, Jr.

Feb. 27, 28	University of Alberta
Mar. 1, 2	University of British Columbia
Mar. 3	Victoria (B.C.) College
Mar. 6, 7, 8, 9, 10	University of Washington and neighboring institutions
Mar. 13, 14	University of Alaska
Mar. 16, 17	Eastern Montana College

Frank M. Stewart

Feb. 2, 3	St. John's University, Jamaica, New York
Feb. 6, 7	Mansfield State College
Feb. 9, 10	Grove City College
Feb. 13, 14, 15	State Teachers College, Indiana, Pennsylvania
Feb. 16, 17	Washington and Jefferson College
Feb. 27, 28	Alfred University
Mar. 1, 2	State U. College of Education, Buffalo
Mar. 3, 4, 5, 6	University of Rochester
Mar. 7, 8	Hobart and William Smith Colleges
Mar. 9, 10	Hamilton College
Mar. 13, 14, 15	The St. Lawrence University, Clarkson College of Technology
Mar. 16, 17, 18	University of Montreal
Apr. 10, 11	Vassar College
Apr. 12, 13	Simmons College
Apr. 14, 15, 16, 17	Tufts University
Apr. 24, 25, 26, 27	Shippensburg State College, Wilson College, Dickinson College
Apr. 28, 29	Montclair State College
May 1, 2	Lafayette College

Robert M. Thrall

Jan. 30, 31	Antioch College
Feb. 2, 3	Hiram College
Feb. 6, 7	Bowling Green State University
Feb. 9, 10	Baldwin-Wallace College
Feb. 13, 14	East Carolina College
Feb. 15	University of North Carolina
Feb. 16, 17	North Carolina State College
Feb. 20, 21, 22	Sweet Briar College, Randolph-Macon Woman's College, Lynchburg College
Mar. 6, 7	University of Richmond
Mar. 9, 10	College of William and Mary
Mar. 13, 14, 15	Florida A. and M. University
Mar. 16, 17	Florida State University
Mar. 21, 22	University of Florida
Mar. 23, 24	Florida Presbyterian College
Apr. 4, 5	University of Miami
Apr. 8	Southeastern Sectional meeting of M.A.A.
Apr. 10, 11	University of Delaware
Apr. 13, 14	West Virginia University
Apr. 17, 18	University of South Carolina
Apr. 20, 21	Agnes Scott College
Apr. 24, 25, 26, 27, 28	University of Georgia, Emory University, Georgia Institute of Technology

May 1, 2	Knoxville College
May 4, 5	Xavier University
May 15, 16	Andrews University
May 18, 19	Western Michigan University
May 22, 23	General Motors Institute
May 25	Michigan State University Oakland

Schedules for W. T. Guy, Jr., and Paul C. Rosenbloom were not available at the time that this note was written.

Plans are being made for a continuance of the Visiting Lecturer Program into 1961–62; announcement will be made about May 1, 1961.

REPORT ON THE PROGRAM OF VISITING LECTURERS TO SECONDARY SCHOOLS, 1959–60

Plan of operation. The program was operated during the academic year, 1959–60, in nine regions: Alabama-Arkansas, California, Connecticut-Maine-Rhode Island-Vermont, Delaware-District of Columbia-Maryland-West Virginia, Idaho-Oregon-Washington, Illinois, Kansas-Missouri-Nebraska, Montana-North Dakota-South Dakota, North Carolina-South Carolina.

TABLE SHOWING NUMBER OF LECTURERS AND SCHOOLS VISITED 1959–60

	Northwest	California	Montana-North Dakota-South Dakota	Kansas-Nebraska-Missouri	Arkansas-Alabama	Illinois	Delaware-District of Columbia-Virginia-West Va.	North Carolina-South Carolina	New England	Total
Number of Mathematicians	26	60	5	6	3	9	9	1	1	120
Number of Schools	90	150	69	51	30	44	73	85	72	664
Number of Days	91	80	45.5	42	28	44	48	54	72	504.5
Number of States	3	2	3	3	3	1	4	2	4	25
Audience	16,800	15,000	9,975	9,900	4,051	9,423	6,500	10,880	10,500	93,029

In addition to members of the Committee on Visiting Lecturers to Secondary Schools, five other mathematicians served as regional representatives. Regional representatives assumed responsibility for regional publicity, obtaining applications from schools in their regions, selecting lecturers from an approved list prepared by the Committee, and scheduling lectures.

The lecturers were prepared not only to give lectures on mathematical topics but to confer with students on future opportunities in study and employment, and discussed teaching problems and curriculum with members of the staff. In short, the lecturers cooperated with the schools in all ways possible toward the furtherance of the aims of the program.

Schools were informed that a maximum daily load for a lecturer would be two lectures to students and one to teachers and that a lesser load would be preferable. The Committee recommended that the lecturer not address a general assembly of students. It was suggested that regional representatives encourage schools to make contributions in partial payment of lecturer's expenses.

Two regions were served by lecturers on leave of absence during the second semester. Professor Israel Rose of the University of Massachusetts lectured in New England states and Professor W. Norman Smith of the University of Wyoming was "full-time" lecturer in North and South Carolina. California and the states of the Northwest made use of a larger number of lecturers than other regions. In general, lecturers in the West Coast regions made trips from their colleges to schools in the neighboring cities and counties. The remaining regions were served by 3 to 9 lecturers, who in a number of instances made trips of a week's duration to several schools in a particular part of a state or region.

Evaluation. The reports from schools show that the lecturers were very well received. Schools in all of the regions asked that the program be continued another year. One regional representative reports that the visits by all but two of the lecturers were unanimously highly praised, and that one of the others apparently improved with time as each report was more favorable than the preceding one. A number of schools in this region reported some regrets for having used the lecturer for speaking to rather large groups of ninth and tenth grade students. Many felt that the lecturer was much more effective in stimulating the advanced students.

A few typical comments from teacher reporters are:

"The lecturer met the students on their level. They were most enthusiastic. His time with us was too short. Could we in the future have a lecturer for two days?"

"Some students came for all four lectures and wished for more."

"The principal thought the visit was very worthwhile."

"All enjoyed it very much. Math has taken on a note of greater importance."

"The math faculty is very happy about the program. The anticipation was good for us and the realization better."

"The lecturer captivated his audience."

Program participants. A partial list of the visiting lecturers includes:

Henry Alder	Wade Ellis	Thurman S. Peterson
Bradford Arnold	Robert E. Gaskell	Pasquale Porcelli
Wilfred Barnes	George Hufford	R. W. Rampfer
Ross A. Beaumont	V. James	L. A. Ringenberg
William E. Briggs	George R. Johnson	S. T. Rio
Z. W. Birnbaum	Burton W. Jones	J. B. Roberts
K. A. Bush	H. S. Kaltenborn	Israel Rose
Donald Bushaw	Antony E. Labarre, Jr.	Paul Rosenbloom
J. R. Byrne	Calvin Long	Hans Sagan
Theodore S. Chihara	Arvid Lonseth	Norman Smith
Leon Cohen	Richard P. Mayer	S. Stein
H. W. Crowley	Rogert E. McKelvey	Alfred Willcox
Mary P. Dolciani	Albert Nijenhuis	William B. Woolf
Roy Dubisch	Cletus Oakley	James H. Zant

The regional representatives for 1959–60 were:

Russell N. Bradt, University of Kansas, Lawrence; John A. Brown, University of Delaware, Newark; Roy Dubisch, Fresno State College; W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts; Harvey M. Gelder, Western Washington College of Education, Bellingham; Adrien L. Hess, Montana State College, Bozeman; Houston T. Karnes, Louisiana State University, Baton Rouge; Thomas D. Reynolds, Duke University, Durham, North Carolina; Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Indiana.

Plans for 1960–61. The program will be carried on during 1960–61 on much the same basis as last year. For this year there are ten regions. The regions, with the names of regional representatives given in parentheses are: Alabama-Florida-Georgia (Houston T. Karnes); California-Nevada (Roy Dubisch); Iowa-Minnesota-Wisconsin (H. Vernon Price); Kentucky-Michigan-Ohio (Marie S. Wilcox); Maryland-Virginia-West Virginia (Malcolm W. Oliphant); New Jersey-Pennsylvania (John A. Brown); New York (John F. Randolph); Idaho-Oregon-Washington (Harvey M. Gelder); Oklahoma-Texas (William T. Guy); Arizona-New Mexico-Utah (Charles Wexler).

The Committee has approved a list of lecturers, including about the same number as last year, and in which there are some duplications with last year and a number of new persons who are assisting with the program. There will be no lecturers during 1960–61 on a leave-of-absence basis. The new program is made possible by a third grant from the National Science Foundation to the Mathematical Association for this purpose. Brochures announcing the program may be obtained from any of the regional representatives. A new member of the committee on visiting lecturers is Professor F. A. Ficken, New York University.

This year, while the committee in general believes that lecturers to general assemblies of students are not to be recommended, it does recognize that in some instances these can be quite successful; hence, a special list of lecturers is being prepared so that if schools make requests for a general assembly speaker, the regular lecturer or a different one may be available.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1451. *Proposed by Anice Seybold, North Central College, Naperville, Illinois*

A student makes an error in breaking a fraction into partial fractions. He writes

$$\frac{x^4 - 3x^3}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

He then clears of fractions and substitutes in succession -1, 1, and 2 as values of x in order to obtain three equations to solve for A, B, and C. Another student correctly carries out the indicated division and uses the remainder, $-x^2-3x+2$, correctly. He writes

$$\frac{-x^2 - 3x + 2}{(x+1)(x-1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Both students get the same values for A, B, and C. How does this happen?

E 1452. *Proposed by N. A. Court, University of Oklahoma*

Find two positive integers such that their sum will be a factor of their product.

E 1453. *Proposed by José Gallego-Diaz, University of Puerto Rico*

Let A be the sum of the digits of a natural number N, let $B=A+N$, let A' be the sum of the digits of the number B, and let $C=B+A'$. Find A if the digits of C are those of A in reverse order.

(Dedicated to the memory of Victor Thébault.)

E 1454. *Proposed by Leonard Carlitz, Duke University*

If Δ, R, r denote the area, circumradius, and inradius of a triangle with sides a_1, a_2, a_3, show that (1) $(R+r)^2 \geqq \Delta\sqrt{3}$, (2) $(a_1a_2a_3)^2 \geqq (4\Delta/\sqrt{3})^3$, (3) $(R\sqrt{3})^3 \geqq a_1a_2a_3$, with equality only when the triangle is equilateral.

E 1455. *Proposed by M. T. L. Bizley, London, England*

Let O be the origin, X the point $(p, 0)$, and Y the point $(0, p)$, where p is a positive prime number. The triangle OXY is divided into p triangles (of equal area) by the lines joining O to the points $(p-r, r)$ for $r=1$ to $r=p-1$. The inte-

riors of the outermost two of these triangles clearly contain no lattice points (*i.e.*, points whose coordinates are both integers). Prove that the interiors of the remaining $p-2$ triangles all contain equal numbers of lattice points.

SOLUTIONS

Two Related Inequalities

E 1421 [1960, 593]. *Proposed by* (1) *S. P. Franklin and G. A. Hutchison, University of California at Los Angeles, and* (2) *W. R. Becker, New York City*

(1) Prove, for any integer $n>1$, $n/(n-1)>\{n^2/(n^2-1)\}^n$.

(2) Prove, for all $a>0$ and $b>0$, $\{(a+1)/(b+1)\}^{b+1}\geqq(a/b)^b$.

Solution by Joe Lipman, Harvard University. By elementary calculus $(a+1)^{b+1}/a^b$, considered as a function of a, has an absolute minimum when $a=b$. This is tantamount to (2), which reduces to (1) when $b=n-1$ and $a=n$.

Also solved by A. N. Aheart, R. H. Anglin, Leon Bankoff, P. R. Chernoff, D. B. Coleman, A. E. Danese, W. G. Dotson, Jr., F. J. Duarte, D. L. Faass, David Friedman, Michael Goldberg, L. D. Goldstone, R. E. Greene, S. H. Greene, R. L. Helmbold, V. E. Hoggatt, J. E. Homer, Jr., A. S. Howard, Erwin Just, Peter Marks, D. C. B. Marsh, C. S. Ogilvy, Thomas Porsching, L. A. Ringenberg, R. E. Shafer, Richard Sinkhorn, D. R. Sondergeld, Guy Torchinelli, W. C. Waterhouse, Alan Wayne, C. C. Yalavigi, David Zeitlin, and the proposers. Late solutions by D. A. Breault, N. K. Govil, and M. A. Malik.

Many solvers obtained (1) from the Bernoulli inequality $(1+x)^n>1+nx$, $x>-1$ and $\neq 0$, by setting $x=-1/n^2$. Marsh obtained (1) by considering, for $n>1$, the n numbers $1-1/n, 1, \cdots, 1$; these are positive and not all equal, whence their geometric mean, $(1-1/n)^{1/n}$, is strictly less than their arithmetic mean, $1-1/n^2$. Duarte and Hoggatt gave noncalculus proofs of (2). Friedman and Zeitlin showed more generally that $[(a+c)/(b+c)]^{b+c}\geqq(a/b)^b$, $a>0$, $b>0$, $c\geqq 0$.

Polynomial Solutions of $f(x^2)+f(x)f(x+1)=0$

E 1422 [1960, 593]. *Proposed by R. W. Kilmoyer, Jr., Lebanon Valley College*

Find polynomials $f(x)$ such that $f(x^2)+f(x)f(x+1)=0$.

Solution by T. K. Cook and C. F. Pinzka, University of Cincinnati. We note that $f(z)=0$ implies $f(z^2)=0$ and $f((z-1)^2)=0$. Since the set of zeros of $f(x)$ is finite and closed under the transformation $z\to z^2$, each z must lie at the origin or on the unit circle of the complex plane. Closure under the transformation $z\to(z-1)^2$ restricts the possible zeros to 0 and 1. Thus $f(x)$ is of the form $ax^m(x-1)^n$, and substitution in the functional equation leads to $f(x)\equiv 0$ or $f(x)=-x^n(x-1)^n$, $n=0, 1, 2, \cdots$.

Aslso solved by A. N. Aheart, Ken Alles, E. J. Barbeau, Jr., Alan Beal, J. L. Brown, Jr., Leonard Carlitz, W. G. Dotson, Jr., Robert Farrell and John Wood (jointly), J. F. Foley, Michael Goldberg, R. E. Greene, S. H. Greene, Erwin Just and Norman Schaumberger (jointly), A. G. Konheim, Joe Lipman, Peter Marks, D. C. B. Marsh, R. W. Means, R. A. Melter, Marvin Mielke, Morris Morduchow, Jack Silver, Richard Sinkhorn, Wu Ta-Sun, Guy Torchinelli, W. C. Waterhouse, C. C. Yalavigi, and the proposer.

Many of the submitted solutions were incomplete. Carlitz remarked that the only polynomial solutions of $f(x^2)+f(x)f(x-1)=0$ are $f(x)\equiv 0$ and $f(x)=-(x^2+x+1)^n$, $n\geqq 0$.

Closed Self-intersecting Curves

E 1423 [1960, 593]. *Proposed by Aboulghassem Zirakzadeh, University of Colorado*

Consider a plane closed curve possessing only ordinary points and a finite number of double points. Assign a positive direction to the curve and, starting from an ordinary point A, trace the curve in the given positive direction. Assign number 1 to the first double point met, 2 to the second double point met, and so on until you come back to A. Prove that of the two integers assigned to any double point, one is always even and the other always odd.

Solution by W. C. Waterhouse, Harvard University. A proof is given in Selection 10 of *The Enjoyment of Mathematics*, by H. Rademacher and O. Toeplitz (translated into English by H. Zuckerman), and is roughly as follows.

We want to show that between successive passages through a given double point an even number of double points are passed through. Call the part of the curve traced (itself a closed curve) B, and the rest of the curve (also a closed curve) C. All double points of B are certainly passed through twice, and we need consider only the intersections of B and C. But C can be replaced by a regular curve without changing its intersections with B, and then the Jordan Curve Theorem shows that there are an even number of intersections of B with C.

Also solved by E. J. Barbeau, Jr., Brother Joseph Heisler, Michael Goldberg, L. D. Goldstone, Peter Marks, D. C. B. Marsh, J. C. Mathews, and Krishna Sarati.

Some of the above solutions were questionable. The problem seems to have arisen in an observation of Gauss, and it has an intimate connection with knot theory.

Complete Sequences

E 1424 [1960, 593]. *Proposed by V. E. Hoggatt and Charles King, San Jose State College*

A sequence $\{W_i\}$ of positive integers is *complete* if for each positive integer N there exists a subsequence $\{W_{i_j}\}_{j=1}^{k}$ of $\{W_i\}$ such that $N=\sum_{j=1}^{k} W_{i_j}$.

(1) Show that the sequence $\{F_i\}$ of Fibonacci numbers ($F_{i+2}=F_{i+1}+F_i$, $F_1=F_2=1$) is complete.

(2) Show that if any one member of the sequence $\{F_i\}$ is deleted, the sequence is still complete.

(3) Show that if two members of the sequence $\{F_i\}$ are deleted, the sequence becomes incomplete.

[*Note.* The problem has been reworded more felicitously than originally.]

Solution by Jack Silver, Montana State University. Let $\{W_i\}$ be obtained from $\{F_i\}$ by deleting one member from $\{F_i\}$. Let N be the first positive integer not representable as the sum of a subsequence of $\{W_i\}$. Either $\sum_{i=1}^{n} F_i$ or $\sum_{i=1}^{n+1} F_i - F_j$, for some $j<n+1$, equals $\sum_{i=1}^{n} W_i$, whence $\sum_{i=1}^{n} W_i \geqq \sum_{i=1}^{n+1} F_i - F_{n+1} = \sum_{i=1}^{n} F_i = F_{n+2}-1$. Then, if n is the greatest k such that $F_k \leqq N$,

$\sum_{i=1}^{n} W_i \geqq N$. Let L be the g.l.b. of the sums of all subsequences of $\{W_i\}$ whose sums exceed N. Say $L=\sum_{j=1}^{k} W_{i_j}$, where the terms of the series are arranged in ascending order of magnitude. If $W_{i_1}=1$, delete it to get $\sum_{j=2}^{k} W_{i_j}=L-1\geqq N$. Otherwise we have $N>W_{i_1}-1\geqq 1$, and, by the defining property of N, $W_{i_1}-1=\sum_{r=1}^{m} W'_r$ where $W'_r<W_{i_j}$ for all r and $j\geqq 2$, whence $\sum_{r=1}^{m} W_{i_r}+\sum_{j=2}^{k} W_{i_j}=L-1\geqq N$. Since in both cases we obtain expansions contradicting the defining properties of L and N, (2) follows.

Since (2) implies (1), (1) also follows.

If both F_k and F_r, $k<r$, are deleted, then $\sum_{i=1}^{r-2} W_i=\sum_{i=1}^{r-1} F_i-F_k=F_{r+1}-1-F_k<F_{r+1}-1=W_{r-1}-1$; thus W_{r-1} cannot be expanded in the required way, and (3) follows.

Also solved by J. L. Brown, Jr., Michael Goldberg, John Jordan, Erwin Just, Joe Lipman, Peter Marks, D. C. B. Marsh, Marvin Mielke, Donna J. Seaman, R. E. Shafer, R. P. Tapscott, Guy Torchinelli, W. C. Waterhouse, and the proposers.

The identity $\sum_{i=1}^{n} F_i=F_{n+2}-1$, employed in the above proof, is easily established by induction. Brown obtained necessary and sufficient conditions for a sequence of positive integers to be complete, and for a complete sequence to remain complete after the deletion of an arbitrary member. The proposers pointed out that the sequence $\{2^i\}$ is complete, but the sequence $\{L_i\}$ of Lucas numbers ($L_{i+2}=L_{i+1}+L_i$, $L_1=1$, $L_2=3$) is incomplete. Deletion of any one member from the first sequence renders it incomplete; addition of the number 2 to the Lucas sequence renders it complete

Square within a Triangle

E 1425 [1960, 593]. *Proposed by D. J. Newman, Brown University*

If a square lies within a triangle, prove that the area of the square does not exceed half the area of the triangle.

Solution by the proposer. By a variational procedure we can see that the largest square in a given triangle has all four vertices on the sides of the triangle. It is trivial to show that three vertices are so located; that the fourth is follows from the easily proven fact that a square not having all four vertices on the sides of the triangle has an infinitesimal motion sending it into the interior of the triangle.

So let the side s of the square lie completely on the side c of the triangle. Draw the altitude h to c; it is cut by the square into lengths $h-s$ and s. By similar triangles we have $(h-s)/s=h/c$, and so $1/s=1/c+1/h$. Hence $1/s^2=(1/c+1/h)^2\geqq 4/ch$, which is to say that the area of the square $=s^2\leqq ch/4$ = half the area of the triangle.

Also solved by Leon Bankoff, William Chapco, T. R. Curry, Guy Di Antonio, Jane Evans, Michael Goldberg, L. D. Goldstone, Ned Harrell, S. L. Hunt, A. R. Hyde, Erwin Just, L. M. Kaplan, Peter Marks, D. C. B. Marsh, Marvin Mielke, Jack Silver, C. C. Yalavigi, and David Zeitlin.

In the inequality $(1/c+1/h)^2\geqq 4/ch$ we have equality when and only when $c=h$, so the area of the square can actually equal half the area of the triangle only when, for appropriate lettering of the triangle, $c=h$ and angles A and B are nonobtuse. Several solvers pointed out that in the problem the word "square" can be replaced by the word "parallelogram," for the one can be carried into the other by an affine transformation, which multiplies all areas by a constant factor. The result

of the problem follows from the stronger theorem: "If F is a convex curve surrounding a closed set K of area A then there is a parallelogram inscribed in F of area $\frac{1}{2}A$. Moreover there is no parallelogram of area strictly larger than $\frac{1}{2}A$ inscribed in F if and only if F is a triangle." See this MONTHLY, vol. 67, March 1960, C. M. Fulton and S. K. Stein, "Parallelograms Inscribed in Convex Curves"

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4947. *Proposed by D. J. Newman, Yeshiva University*

Let $a_{mn}=\exp(-m-n/e^{2m})$. Prove that the infinite matrix $A=(a_{mn})$ represents a bounded operator on Hilbert space. That is, it takes vectors $x_1, x_2, \cdots$ for which $\sum x_n^2<\infty$ into vectors $y_1, y_2, \cdots$ for which $\sum y_n^2<\infty$.

4948. *Proposed by Lawrence Glasser, Carnegie Institute of Technology*

Show that the volume of any sphere in Hilbert space is zero.

4949. *Proposed by Richard Bellman, the RAND Corporation*

Consider the problem of maximizing the functional

$$J(\lambda_1, \cdots, \lambda_n) = \int_0^T \left(\sum_{i=1}^n a_i\lambda_i(t) \right) \exp\left(\int_t^T \sum_{i=1}^n b_i\lambda_i(s)ds \right) dt,$$

over all functions $\lambda_i(t)$ satisfying the conditions $\lambda_i(t)\geqq 0$, $\sum_{i=1}^n \lambda_i(t)=1$, $0\leqq t\leqq T$. Show that the solution can be obtained in terms of the solution of the differential equation $du/dt=\max_{1\leq i\leq n}(a_i+b_iu)$, $u(0)=0$, and thus determine the solution.

4950. *Proposed by T. S. Nanjundiah, Central College, Bangalore, India*

Evaluate

$$\prod_{n=1}^{\infty}{}' \frac{n^3+p^3}{n^3-p^3}, \qquad \sum_{n=1}^{\infty}{}' \frac{n^2}{n^6-p^6}, \qquad p=1, 2, \cdots,$$

the accent indicating that $n=p$ is to be omitted.

The product is known for the case $p=1$. See Bromwich, *Infinite Series* (1926), p. 313, Ex. 18; *Collected Papers of Ramanujan*, (1927), p. 322, Qn. 261.

4951. *Proposed by J. F. Heyda, General Electric Co., Cincinnati, Ohio.*

Solve the nonlinear integral equation

$$mx + b = \int_0^x \frac{dt}{\sqrt{\{P(t) - P(x)\}}}, \qquad x \geqq 0, m \geqq 0, b \geqq 0,$$

for $P(0) - P(x)$, where $P(x)$ is monotone decreasing.

4952. *Proposed by Lawrence Shepp, Princeton University*

Banach has shown (*Opérations Linéaires*, pp. 29–34) that there exists a method which satisfies certain natural axioms whose domain is the set of bounded sequences. His proof is unnecessarily long because he uses the functional:

$$P(S_n) = \operatorname*{glb}_{\{i_1, i_2, \cdots, i_k\}} \limsup_n \sum_{j=1}^{k} \frac{S_{n+i_j}}{k} \text{ for bounded } \{S_n\},$$

where the same proof works for the simpler

$$q(S_n) = \operatorname*{glb}_k \limsup_n \sum_{j=1}^{k} \frac{S_{n+j}}{k}.$$

Prove:

$$P(S_n) = q(S_n) = \lim_k \limsup_n \sum_{j=1}^{k} \frac{S_{n+j}}{k}.$$

SOLUTIONS

A Function Schlicht in the Upper Half-plane

4890 [1960, 187]. *Proposed by H. S. Shapiro, New York University*

Let $f(t)$ be positive, integrable and strictly decreasing in $(0, \infty)$, and suppose that the Fourier cosine transform of f is strictly decreasing in $(0, \infty)$. Then $F(z) = \int_0^\infty f(t)e^{izt}dt$ is regular and schlicht in the upper half-plane.

Indications of proof by the proposer. An elementary topological argument reduces the problem to showing that $F(z)$ cannot take the same value at two distinct points of the real axis. (Note that analyticity of $F(z)$ follows from uniform convergence by a familiar argument.) Now, if $F(x_1) = F(x_2)$, the Fourier cosine transform $\phi_c(x)$ of f must take the same value at x_1 as at x_2. This implies, since ϕ_c is even and decreasing in $(0, \infty)$, that $x_1 = -x_2 \neq 0$. But also the sine transform $\phi_s(x)$ must take equal values at x_1 and x_2, *i.e.*,

$$\phi(x_1) = \phi(x_2) = \phi(-x_1) = -\phi(x_1)$$

and so $\phi(x_1) = 0$. But because $f(t)$ is $\downarrow$, its sine transform can vanish only at $x = 0$ (Titchmarsh, *The Fourier Integral*, Theorem 123; he proves $\phi_s \geqq 0$, but the strong statement follows if one assumes f is strictly decreasing).

A and B, where A^2+B^2 is prime

4893 [1960, 294]. *Proposed by Joe Lipman, University of Toronto*

Let A, B be positive integers, A odd, B even, and let $A^2+B^2=p$, a prime,

(a) Show that, except perhaps for sign, A and B are respectively the real and the imaginary parts of S, where

$$S = \sum_{x=0}^{\frac{1}{2}(p-1)} \exp\left\{\tfrac{1}{2}\pi i \operatorname{ind}(x^4 + a^4)\right\},$$

with a any nonzero, fixed quadratic residue (mod p).

(b) Let R be any fixed quadratic residue (mod p) and N any fixed nonresidue. Prove

$$A = \sum_{x=0}^{\frac{1}{2}(p-1)} \left(\frac{x^4 + R}{p}\right), \qquad B = \sum_{x=1}^{\frac{1}{2}(p-1)} \left(\frac{x^4 + N}{p}\right)$$

where the summands are Legendre symbols.

Solution by Leonard Carlitz, Duke University. (a) Let g be a primitive root (mod p), where $p\equiv 1 \pmod 4$ and let C_i denote the set of numbers (mod p), $\{g^{i+4r}\}$, $(r=0, 1, \cdots, \frac{1}{4}(p-5))$. Also let (i, j) denote the number of solutions of

$$\alpha_i + 1 \equiv \alpha_j \pmod p \qquad (\alpha_i \in C_i,\ \alpha_j \in C_j).$$

Gauss (*Untersuchungen über höhere Arithmetik, Theorie der biquadratischen Reste I*, pp. 511–533) determined the numbers (i, j). For $p\equiv 1 \pmod 8$ he found

$$16(0, 0) = p - 6A - 11, \qquad 16(0, 1) = p + 2A - 4B - 3,$$
$$16(0, 2) = p + 2A - 3, \qquad 16(0, 3) = p + 2A + 4B - 3;$$

while for $p\equiv 5 \pmod 8$,

$$16(0, 0) = p + 2A - 7, \qquad 16(0, 1) = p + 2A + 4B + 1,$$
$$16(0, 2) = p - 6A + 1, \qquad 16(0, 3) = p + 2A - 4B + 1.$$

On the other hand

$$S = \sum_{\substack{x=0 \\ x^4 \not\equiv -1}}^{\frac{1}{2}(p-1)} i^{\operatorname{ind}(x^4+1)} = 1 + 2 \sum_{\substack{\alpha \in C_0 \\ \alpha \not\equiv -1}} i^{\operatorname{ind}(\alpha+1)}.$$

When $p\equiv 1 \pmod 8$ we get

$$S = 1 + 2\{(0, 0) + (0, 1)i - (0, 2) - (0, 3)i\} = -A - Bi.$$

When $p\equiv 5 \pmod 8$ we get

$$S = 1 + \tfrac{1}{8}\{(0, 0) + (0, 1)i - (0, 2) - (0, 3)i\} = A + Bi.$$

In either case we have $p=A^2+B^2$, $A\equiv 1 \pmod 4$.

Since

$$\sum_{x=0}^{\frac{1}{2}(p-1)} i^{\mathrm{ind}(x^4+a^4)} = 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\mathrm{ind}(x^4+a^4)} = 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\mathrm{ind}(a^4x^4+a^4)}$$

$$= 1 + \frac{1}{2}\sum_{x=1}^{p-1} i^{\mathrm{ind}(x^4+1)} = \sum_{x=0}^{\frac{1}{2}(p-1)} i^{\mathrm{ind}(x^4+1)},$$

we may assume $a=1$; in fact, it is not necessary to assume that a is a quadratic residue of p.

(b) Put

$$\psi(r) = \sum_{x=0}^{\frac{1}{2}(p-1)} \left(\frac{x^4+r}{p}\right),$$

so that

$$2\psi(r) - 1 = \sum_{x=0}^{p-1}\left(\frac{x^4+r}{p}\right) = \sum_{y=0}^{p-1}\left(1+\left(\frac{y}{p}\right)\right)\left(\frac{y^2+r}{p}\right)$$

$$= -1 + \sum_{y=0}^{p-1}\left(\frac{y}{p}\right)\left(\frac{y^2+r}{p}\right);$$

where we have used the formula

$$\sum_{y=0}^{p-1}\left(\frac{y^2+r}{p}\right) = -1 \qquad (r \not\equiv 0 \pmod{p}).$$

Thus

$$(*) \qquad 2\psi(r) = \sum_{y=0}^{p-1}\left(\frac{y}{p}\right)\left(\frac{y^2+r}{p}\right).$$

We have next

$$4\sum_{r=0}^{p-1}\psi^2(r) = \sum_{x,y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\sum_{r=0}^{p-1}\left(\frac{x^2+r}{p}\right)\left(\frac{y^2+r}{p}\right)$$

$$= (p-1)\sum_{x^2\equiv y^2}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right) - \sum_{x^2\not\equiv y^2}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)$$

$$= p\sum_{x^2\equiv y^2}\left(\frac{xy}{p}\right) - \sum_{x,y}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right) = 2p(p-1).$$

Since $\psi(rc^2) = (c/p)\psi(r)$, this reduces to

$$(**) \qquad 2(p-1)(\psi^2(R)+\psi^2(N)) = 2p(p-1),\ \psi^2(R)+\psi^2(N) = p.$$

Since

$$\psi(1) = \sum_{y=1}^{\frac{1}{2}(p-1)} \left(\frac{y}{p}\right)\left(\frac{y^2+1}{p}\right)$$

and $y^2+1 \equiv 0$ for exactly one y in $1 \leqq y \leqq \frac{1}{2}(p-1)$, it follows that $\psi(1) \equiv \frac{1}{2}(p-3) \equiv 1 \pmod 2$.

The formula (**) with $\psi(r)$ defined by (*) is due to Jacobsthal (*Journal für die reine und angewandte Mathematik*, vol. 132 (1907), pp. 238–245).

Also solved by Emma Lehmer and the proposer.

An Inequality for Hermitian Matrices

4894 [1960, 294]. *Proposed by Olga Taussky, California Institute of Technology*

Let A, B be two positive definite Hermitian matrices which can be transformed simultaneously by unitary transformation to diagonal forms of similarly ordered numbers. Let x be any vector of complex numbers. Show that $(Ax, x)(Bx, x) \leqq (ABx, x)(x, x)$, and discuss the case of equality. Two sets of n positive numbers, $\{a_i\}$, $\{b_i\}$ are called similarly ordered if $(a_i - a_k)(b_i - b_k) \geqq 0$ for all $i, k = 1, \cdots, n$.

Solution by Marvin Marcus, University of British Columbia. It will be seen that the positive definite condition is unnecessary. We assume that A and B are commutative Hermetian matrices with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_n$ and $\mu_1 \geqq \cdots \geqq \mu_n$ respectively. The conditions of the problem imply that there exists an orthonormal basis of common eigenvalues of A and B, $e_1, \cdots, e_n$, such that $Ae_i = \lambda_i e_i$, $Be_i = \mu_i e_i$, $i = 1, \cdots, n$. For any vector x let p_x and q_x be respectively the smallest and largest integers i for which $\sigma_i = |(x, e_i)|^2 \neq 0$. We prove the

THEOREM. *$(x, x)(ABx, x) - (Ax, x)(Bx, x) \geqq 0$ with equality if and only if either $\lambda_{p_x} = \lambda_{q_x}$ or $\mu_{p_x} = \mu_{q_x}$. If equality holds for an x for which $p_x = 1$ and $q_x = n$ then either A or B is a multiple of the identity.*

Proof. We compute

$$\begin{aligned}(x, x)(ABx, x) - (Ax, x)(Bx, x) &= \sum_{i=1}^{n} \sigma_i \sum_{i=1}^{n} \sigma_i \lambda_i \mu_i - \sum_{i=1}^{n} \sigma_i \lambda_i \sum_{i=1}^{n} \sigma_i \mu_i \\ &= \sum_{i,j=1}^{n} (\sigma_i \sigma_j \lambda_j \mu_j - \sigma_i \sigma_j \lambda_i \mu_j) = \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i \sigma_j (\lambda_i - \lambda_j)(\mu_i - \mu_j).\end{aligned}$$

Since $\sigma_i \geqq 0$ and $(\lambda_i - \lambda_j)(\mu_i - \mu_j) \geqq 0$ the inequality follows (actually this is the Tchebycheff inequality). The case of equality can occur if and only if $\sigma_i \sigma_j (\lambda_i - \lambda_j)(\mu_i - \mu_j) = 0$ for all $1 \leqq i < j \leqq n$. Setting $i = p_x$, $j = q_x$ and noting that $\sigma_{p_x} \sigma_{q_x} > 0$ we have at least one of the possibilities $\lambda_{p_x} = \lambda_{q_x}$ or $\mu_{p_x} = \mu_{q_x}$. On the

other hand if either of these possibilities obtain then for any $1 \leqq i < j \leqq n$ either (a): $i < p_s$ or $j \geqq q_s$ and $\sigma_i\sigma_j = 0$, or (b): $p_s \leqq i < j \leqq q_s$ and $(\lambda_i - \lambda_j)(\mu_i - \mu_j) = 0$ follows from the ordering of the eigenvalues. Hence we conclude that

$$\sum_{i,j=1}^{n} \sigma_i\sigma_j(\lambda_i - \lambda_j)(\mu_i - \mu_j) = 0$$

and the equality holds. The last statement follows from the fact that a Hermitian matrix with all eigenvalues equal must be a multiple of the identity.

Also solved by A. F. Kaupe, Jr., R. F. Rinehart, J. E. Potter, and the proposer.

The Laplace Transform of $\log^2 x$

4895 [1960, 295]. *Proposed by E. S. Keeping, University of Alberta*

γ being Euler's constant, prove that

$$\int_0^\infty (\log x)^2 e^{-x} dx = \tfrac{1}{6}\pi^2 + \gamma^2.$$

Note by T. C. Brown, Reed College, Portland, Oregon. This is problem 3766, for which solutions are found in this MONTHLY 1938, 57–58. Later [1958, 695], M. E. Levenson extended the result to a recursion formula for $\int_0^\infty (\log x)^{n+1} e^{-x} dx$.

Also solved by A. N. Aheart, A. R. Bradley, D. A. Breault, J. W. Brown, R. G. Buschman, L. Carlitz, P. R. Chernoff, P. L. Chassin, A. E. Danese, G. Di Antonio, A. B. Farnell, James Foster, Roberto Frucht, Todd Gitlin, M. L. Glasser, George Glauberman, S. H. Greene, S. W Greenhouse, Hur-bun Hou, M. S. Klamkin, T. V. Lakshminarasimhan, Joe Lipman, Y. Matsuoka John Moon and Ted Petrie, C. S. Ogilvy, J. E. Potter, G. E. Raynor, M. B. Ritterman, P. G. Rooney, P. R. Sanders, R. E. Shafer, George Shortley, F. C. Smith, Dmitri E. Thoro, Harry Weingarten, J. S. White, David Zeitlin, and the proposer—many of whom referred to the earlier problem which your Editor had overlooked.

Singularities on the Unit Circle

4896 [1960, 295]. *Proposed by H. S. Shapiro, New York University*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where the a_n lie in the interior of a Jordan curve Γ in the complex plane, the origin being exterior to Γ. Prove that there exist singularities $z_1, \cdots, z_k$ $(k \geqq 1)$ on the unit circle, and positive integers $n_1, \cdots, n_k$ such that $z_1^{n_1} \cdots z_k^{n_k} = 1$.

Solution by D. J. Newman, Yeshiva University. We need first the

LEMMA. If Γ is a Jordan curve with 0 in its exterior, then there is a polynomial $P(z)$ with $P(0) = 0$ and $\operatorname{Re} P(z) > 1$ for all z in the interior of Γ.

The result follows easily from this lemma, namely $P(a_n)$ is bounded and so $\sum P(a_n)z^n$ is convergent for $|z| < 1$. Since $\operatorname{Re} P(a_n) > 1$ it follows that $\operatorname{Re} P(a_n)z^n \to \infty$ as $z \to 1^-$ and so $z = 1$ is a singularity of $\sum P(a_n)z^n$; hence, for some $k > 0$, $\sum a_n^k z^n$ has a singularity at $z = 1$. But $\sum a_n b_n z^n$ can have singularities only at $z_1 z_2$ where z_1 is a singularity of $\sum a_n z^n$, z_2 a singularity of $\sum b_n z^n$, hence $\sum a_n z^n$

must have k singularities on $|z|=1$ such that $1=z_1z_2\cdots z_k$ and this is the result. (These z_i need not be distinct.)

Proof of lemma. Let a be a large positive number, then $1+a/z$, being analytic inside and on Γ, can be approximated by polynomials. So let $Q(z)$ be such that $|1+a/z-Q(z)|<1$ for z inside Γ. Choose $P(z)=zQ(z)$, then $P(0)=0$ and $\operatorname{Re} P(z)\geqq\operatorname{Re}(z+a-|z|)$ throughout the interior of Γ. If a is large enough, however, the right side is >1.

Also solved by J. L. Ullman and the proposer.

Polynomial in Two Variables

4897 [1960, 295]. *Proposed by D. J. Newman, Yeshiva University*

Let $F(x, y)$ be such that: for any fixed y. $F(x, y)$ is a polynomial in x, and for any fixed x, $F(x, y)$ is a polynomial in y. Prove $F(x, y)$ is a polynomial in x and y.

Editorial Note. The solution is given in a Mathematical Note by F. W. Carroll, *A polynomial in each variable separately is a polynomial*, this MONTHLY, vol. 68, 1961, p. 42. This note was submitted to the Editors before the present problem appeared in this department. Morris Morduchow treated the same problem in: *On surface-fitting in three variables*, Journal of Applied Physics, 20 (1949), pp. 390–392.

Independent solutions submitted by L. Carlitz, P. R. Chernoff, G. Di Antonio, N. J. Fine, Harley Flanders, Todd Gitlin, A. P. Hillman, Joe Lipman, Christopher Metz, J. V. Whittaker, and the proposer. Late solution by J. L. Brown, Jr.

Number of Sequences of Special Type

4898 [1960, 295]. *Proposed by R. C. Reed, University College of the West Indies*

A sequence of n digits, each digit being a 1, 2 or 3, will be called an A_n-sequence. The six permutations of 123 are the only proper A_3-sequences, and for $n>3$ a proper A_n-sequence is defined recursively as follows: an A_n-sequence ($n>3$) is proper if, and only if, it can be obtained from some proper A_{n-1}-sequence by deleting a digit of the A_{n-1}-sequence and putting in its place the other two digits (in either order). Thus the sequences 213, 2233, 31233 are examples of proper A_n-sequences.

Show that the number of distinct proper A_n-sequences is $\frac{3}{4}\{3^{n-1}-2-(-1)^n\}$.

Solution by A. P. Hillman, University of Santa Clara. Define an A_n-sequence to be an S_n if each of 1, 2, 3 appears an odd number of times when n is odd, and if at least two of 1, 2, 3 are present and each appears an even number of times when n is even. If $n>3$, an S_n has one of 1, 2, 3 appearing at least twice. One of these multiple appearances must be next to a different digit. If these distinct adjacent digits are replaced by the third digit, an S_{n-1} results. Induction now shows that an A_n-sequence is proper if and only if it is an S_n.

From one S_n many S_{n+1}'s may be obtained by the given rule. Of these, just three may be obtained as follows: Choose one of 1, 2, 3; place it in front of the S_n; and change the first of the non-chosen digits appearing in the S_n to the other

nonchosen digit. For example, 123 leads to 1133, 2323, and 3223. On the other hand, by the inverse process, corresponding to any S_{n+1} there is a unique S_n with the exception of the six S_{n+1}'s of the form $abcc \cdots c$ if n is even. Hence, this new operation applied to all S_n results without repetition in all S_{n+1} if n is odd, and in all but six of the S_{n+1}'s if n is even. It follows that the number p_n of S_n's satisfies

$$p_{2k} = 3p_{2k-1}, \qquad p_{2k+1} = 3p_{2k} + 6.$$

The desired formula follows immediately by induction.

Also solved by W. J. Blundon, Donald W. Brown, E. N. Gilbert, John B. Kelly, James Singer, and the proposer.

Editorial Note. In terms of the greatest integer function the number of proper A_n-sequences may be written as $3[3^{n-1}/4]$. Several solvers pointed out that the desired number, when n is odd, equals the coefficient of x^n in the expansion of $\sinh^3 x$; when n is even, it is the coefficient of x^n in $\cosh^3 x$ except that this includes the three improper sequences having a single digit repeated.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematical Methods of Operations Research. By Thomas L. Saaty. McGraw-Hill, New York, 1959. $10.00.

The subject of operations research is insufficiently mature to be treated axiomatically, as one can treat probability theory or thermodynamics. The subject can hardly be treated even systematically—since each professional would have a different system. Most books on operations research use the approach of case studies of specific projects with the apparent aim of getting the general idea across. This book is refreshingly different in that an attempt has been made to discuss in considerable detail some of the new branches of applied mathematics which have developed because of and for O.R. There are chapters on linear and quadratic programming, game theory, queueing theory, and optimization. There are also chapters which develop the necessary ideas in probability, statistics, and elementary analysis. The book should be excellent as a text for a senior or first-year graduate course.

WILLIAM VIAVANT
University of Oklahoma

Reduktionstheorie des Entscheidungsproblems im Prädikatenkalkül der Ersten Stufe. By János Surányi. Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1959. 216 pp. About $6.25.

This excellent work is the first systematic presentation of all the main reductions of the decision problem for the predicate calculus. A *reduction class* is a class of formulas such that, for every formula A of the predicate calculus, there can be effectively constructed a formula B in the class such that either A and B are both satisfiable or neither is satisfiable. The following, in particular, are shown to be reduction classes: (1) The class of closed formulas of the form $\forall_{x_1}\forall_{x_2}\forall_{x_3}\mathbf{M}_1 \;\&\; \forall_{y_1}\forall_{y_2}\exists_{y_3}\mathbf{M}_2$, where the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are constructed using only truth functions, individual variables, one-place predicate variables, and at most one two-place predicate variable; (2) The class of closed prenex formulas with a prefix of the form $\exists\forall\exists\forall^2$ in which one-place and at most four two-place predicate variables appear; (3) The class of closed prenex formulas with a prefix of the form $\exists\forall\exists\forall^n$ in which only a single two-place predicate variable appears.

This last reduction class is obtained through the use of arithmetic methods, with the help of the Löwenheim-Skolem theorem. As the author himself noted, the use of the axiom of choice in the proof of that theorem can be avoided by more complex constructions.

Surányi also sketches Kalmár's reduction of the decision problem to the case of arbitrary finite domains (using the Skolem-Herbrand theorem, proved in the appendix) and his proof of Church's theorem on the recursive unsolvability of the general decision problem.

An introductory chapter contains all the information on the predicate calculus which would be necessary for any mathematician who is not primarily a logician to understand the remainder of the book. Before each complicated proof is attempted, the idea underlying it is presented. False starts are sometimes made, followed by explanations of why they won't succeed. The proofs themselves are remarkably complete and very clear. This is an altogether admirable production.

JOSEPH S. WHOLEY
Harvard University

Differential Geometry. By Erwin Kreyszig. University of Toronto Press, Toronto, 1959. xiv+352 pp. $8.50.

An introduction to the differential geometry of curves and surfaces in three-dimensional Euclidean space is set forth here, and the presentation affords excellent preparation for Riemannian geometry of n dimensions. Full use of the tensor calculus is made, and the concepts of tensor analysis are developed as needed in a simple and natural manner. For instance, the Christoffel symbols are not introduced until they are needed (p. 140) to portray the Gauss formulae. This is typical of the author's concern for clarity.

In the preface, the author says that he has tried "to present the whole subject-matter in the simplest possible form consistent with the needs of mathematical rigour." This reviewer is so impressed with the success achieved toward the stated goal, that he would recommend the book for students who are interested in tensor methods and who may not have the benefit of an instructor.

After a useful introductory chapter on preliminaries the second chapter covers the usual theory of space curves but with more than usual lucidity and rigor. The next three chapters introduce foundations of tensor calculus along with surface theory. Pertinent remarks indicate that surface area introduces difficulties not encountered in finding arc length. Chapter six on mappings is an excellent one—unusually complete with introduction of the Bergman metric. In the seventh chapter on absolute differentiation and parallel displacement one sees again how the author prepares the reader by giving continuity, a preview of what is to follow, and by displaying the dominant concepts in a striking way. The last chapter on special surfaces includes minimal surfaces and a treatment on modular surfaces of analytic functions.

One does not find here long lists of exercises at the ends of chapters. Rather, a few problems are strategically located to improve understanding, and the solutions of these appear at the end of the book.

This is a free translation of the author's *Differentialgeometrie* which appeared in the series Mathematik und ihre Anwendungen in Physik und Technik (series A, Vol. 25).

C. E. SPRINGER
The University of Oklahoma

Plane Trigonometry. By Nathan O. Niles. Wiley, New York, 1959. xi+234 pp. $3.95.

This textbook contains twelve chapters pertaining to the following topics in the order indicated: fundamental concepts, trigonometric functions of angles, the right triangle, trigonometric functions of real numbers, fundamental identities, variations and graphs of the trigonometric functions, trigonometric functions of composite angles, logarithms, oblique triangles, inverse trigonometric functions, trigonometric equations, vectors and complex numbers.

Radians are introduced in Chapter 1 and used throughout the text. Another excellent feature of this text is its treatment of the trigonometric functions of real numbers. The discussion of Chapter 4 gives many details and is followed in later chapters by frequent reminders of the two interpretations of the argument of a trigonometric function. It should be noted that functions of the general angle are discussed in Chapter 2 and that the special definitions of the functions of an acute angle are emphasized in Chapter 3. Slide-rule solutions of triangles are discussed in detail.

The following tables are included: four-place values of trigonometric functions of angles in degrees and radians; four-place values of trigonometric functions of real numbers or angles in radians and degrees; four-place logarithms

of numbers from 1–10; four-place logarithms of trigonometric functions of angles in degrees.

The exposition of the text is at the right level for the student. Several illustrations of each topic, diagrams, and answers for odd-numbered problems help to make it a useful text for the student.

This book is adequate for a three-hour, one-semester course in trigonometry. Instructors who wish to put less emphasis upon the solution of triangles will find sufficient material for a two-hour course stressing analytic trigonometry. The terminology and coverage of analytic trigonometry are consistent with recommendations of recent national committees and provide the student with a good foundation for subsequent courses in mathematics.

EDITH R. SCHNECKENBURGER
University of Buffalo

Plane Trigonometry. By A. W. Goodman. Wiley, New York, 1959. Including tables, xvii+267 pp. $4.50. Without tables, xvii+197 pp. $3.75.

This textbook contains an excellent development of the traditional subject matter of trigonometry. A notable feature of the text is its emphasis upon definition, theorem, and proof. In Chapter 0 we find discussions of mathematical symbols and of the meaning of *axiom* and *theorem*. In subsequent chapters definitions are carefully labeled and stated with precision. Statements requiring proof, such as the many formulas of trigonometry, are called theorems and proofs are given. The duality principle pertaining to identities is a new theorem which provides an illustration of one way of extending mathematics.

The book emphasizes solution of triangles as well as analytic trigonometry. The author discusses the trigonometric functions of an acute angle in Chapter 1 so that the student can observe the process of modifying a definition to fit a new situation when he studies the trigonometric functions of a general angle. The following topics follow Chapter 1 in the order indicated: logarithms, logarithmic solution of right triangles, trigonometric functions of a general angle, elementary trigonometric identities, oblique triangles, addition formulas, radian measure, graphs of trigonometric functions, trigonometric equations, inverse trigonometric functions, areas, vectors, complex numbers.

The following tables are available: squares and square roots; values of trigonometric functions to four places; mantissas of common logarithms of numbers to five decimal places; logarithms of trigonometric functions to five decimal places.

The discussion of objective at the beginning of each chapter, clarity and level of exposition, and illustrative problems make this a text which can be read by the student with understanding. The informal style of the author and the format of the book will be appreciated by students. Answers are given for the odd-numbered problems. The *Preface to the Student* contains good suggestions for studying mathematics.

There is sufficient material for a three-hour, one-semester college course for

students who have had no previous work in trigonometry. In view of the rapidly changing curriculum in secondary schools it is probable that many instructors will prefer to use the text for a two-hour, one-semester course in analytic trigonometry or for a trigonometry unit in a freshman course covering various subjects. The book is also suitable for a high school trigonometry course.

The student who uses this text will acquire insight into the nature of mathematics and how it has developed, along with a knowledge of trigonometry.

EDITH R. SCHNECKENBURGER
University of Buffalo

Theory and Solution of Ordinary Differential Equations. By Donald Greenspan. Macmillan, New York, 1960. viii+148 pp. $5.50.

If you have grown weary teaching the traditional "cook-book" course in differential equations, this is the book to consider. The usual content of a differential equations course—minus applications— is presented in a rigorous setting, requiring the student to exhibit understanding of point set theory and elementary real variable theory (in the text and in the variety of exercises). Two proofs of the fundamental existence theorem are presented—one employing the classical Picard iterative technique and the other an elegant approach through the use of functional analysis. In general, the book has a systematic and well-written approach to its subject. In particular, the Frobenius method for solution in series is admirably treated.

I have some reservations, however. I gather the impression from certain topics scantily outlined (special functions, Laplace transform, Sturm-Liouville theory) that the author is pressed for space. Yet he presents linear differential equations of order two in one chapter and then generalizes them practically word for word to those of order n in the next chapter. Also, it is disturbing to find the author recalling the properties of determinants, but neglecting to state theorems involving interchange of limit operations. I might add that the latest editions (Agnew, Churchill, Zygmund) should be cited in the bibliography.

Nevertheless, there is a great deal to praise in this book and it should prove a welcome relief for those who are interested in more than routine solution of differential equations without resort to treatises.

ARTHUR E. DANESE
Union College, Schenectady, N. Y.

A Primer of Real Functions. By Ralph P. Boas, Jr. The Mathematical Association of America (Carus Monograph No. 13), 1960. xi+189 pp.

This excellent two-chapter little book treats a variety a specialized subjects in lively manner. The subject matter, for the most part, is selected from the foundations of analysis and ranges from important pathological examples to fundamental theorems and some of their applications. The first chapter presents some basic notions inherent to the foundations of analysis, and the last chapter treats some important properties of various classes of functions. Of

particular interest here is the treatment of the linear function and convex functions. The entire presentation is (as the author states in his preface) informal and includes some sprinkled bits of wit and philosophy.

PASQUALE PORCELLI
Louisiana State University

Calculus of Functions of One Argument with Analytic Geometry and Differential Equations. By Edward J. Cogan, Robert Z. Norman and Gerald L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1960. x+587 pp. $8.75.

This book is suitable for an introductory course in the study of functions of one variable. Ideas of set theory and logic are introduced early; parts of analytic geometry and trigonometry are presented as they are needed. Material intended for a first semester includes differentiation and integration of exponential and logarithmic functions and some work in differentiation equations. The rest of the book includes trigonometric and inverse trigonometric functions, methods of integration, improper integrals, infinite series and differential equations. The discussion of vectors, postponed until the last chapter, is brief. There is no work on multiple integration. No tables are included but practice in their use is provided through references to Cogan and Norman: *Handbook of Calculus, Difference and Differential Equations.*

This book is well organized and well written; it has many illustrative examples and many problems. The concepts involved are clearly presented. Since students will later use books with standard notation, one questions the advantage of some of the symbols used; *e.g.*, $(x^1[x^2])(4)$ for $f[g(4)]$, with $f(y)=y^1$ and $y=g(x)=x^2$, and the use of ln, sin, and ff to replace $\ln x$, $\sin x$ and $\int f(t)dt$.

FLORENCE M. MEARS
The George Washington University

Advanced Algebra, Part I. By E. A. Maxwell. Cambridge University Press, New York, 1960. ix+311 pp. $2.75.

This book deals with the usual topics of advanced algebra,—polynomials and related equations and inequalities, simultaneous equations, determinants, complex numbers, partial fractions, graphs of rational functions, permutations and combinations, the binomial theorem, summation of finite series, and infinite series. Partial fractions are studied in logical detail for both repeated linear and quadratic factors of the denominator. Methods for summing generalized arithmetic and harmonic series are provided. The binomial series lead to expansions of rational functions in power series, these, in turn, to exponential and logarithmic series. To discuss the latter series the author employs elementary calculus but of necessity states the basic assumptions about convergent series; the detailed discussion might better be postponed to a calculus text.

The book is carefully prepared and concisely written. The author emphasizes the logical structure of algebra as much as he believes consistent with the mathe-

matical maturity of the students for whom the text is intended. His illustrative material contains techniques for solving routine problems and develops methods which the intelligent student can adapt to the solution of numerous challenging examples. The reader must attend closely both to the theory and the illustrative examples if he is to solve the problems of the text. For such a reader the book contains much of immediate interest and provides valuable training for mathematical growth.

HELEN G. RUSSELL
Wellesley College

Geometry. By Charles F. Brumfiel, Robert E. Eicholz, and Merrill E. Shanks. Addison-Wesley, Reading, Mass., 1960. xi+288 pp. $4.75.

The authors present a fairly elementary but mathematically sound (after Hilbert) version of Euclidean geometry. This book is possibly definitive and deserves the most careful consideration.

Chapter 1 is a summary of the student's previous experience with geometry. Chapter 2, Logic, introduces terms of elementary logic, and nature of proof. The vocabulary of logic is not employed much in the sequel. In Chapters 3–6, the major innovations become manifest, and merits of the text's construction appear. The language is formidable, so that one viewing this text for adoption should work most of the problems.

However, the main body of theorems, chapters 7–13, follow nicely. Space Geometry, chapter 14, is not much more than a list of space postulates and theorems, and should be accessible to students who had mastered the course thus far. By now it must be evident that the book is not intended for weak classes. Chapter 15, Analytic Geometry, approximates in content that of a good intermediate algebra, although it is more sound.

The authors of this important and valuable text should be praised. Still, it has defects, mostly of writing, which could prevent extensive adoption. One hopes that it will have enough commercial success to carry it through at least one revision.

Possibly the major contribution of this text will be in helping to revitalize secondary-school geometry.

E. L. WALTERS
William Penn Senior High School
York, Pennsylvania

Dynamic Programming and Markov Processes. By Ronald A. Howard. Technology Press of M.I.T. and Wiley, New York, 1960. viii+136 pp. $5.75.

The motivation of this monograph is best described in the author's own words: "The systems engineer or operations researcher is often faced with devising models for operational systems. The systems usually contain both probabilistic and decision-making features, so that we should expect the resultant model to be quite complex and analytically intractable. This has indeed been the case

for the majority of models . . . proposed. The exposition of dynamic programming by Richard Bellman [*Dynamic Programming*, Princeton University Press, 1957, Chapter XI] gave hope to those engaged in the analysis of complex systems, but this hope was diminished by the realization that more problems could be formulated by this technique than could be solved. Schemes that seemed quite reasonable often ran into computational difficulties . . . not easily circumvented."

The author develops an analytic structure for a decision-making system based on the Markov process as system model. The method of optimization is an iterative technique similar to dynamic programming. Beginning with a description of discrete-time Markov processes, the model is generalized to include economic rewards and the decision process itself. Decision processes with simple probabilistic structures are solved by the policy-iteration method and detailed applications are given to problems of taxicab operation, baseball, and automobile replacement. The succeeding chapters deal with more complicated probabilistic structures (multiple-chain processes), the effect of discounting future rewards, and the analysis of continuous-time Markov processes. The development is swift, concise and lucid. This book is a welcome addition to the growing literature on sequential problems.

H. KAUFMAN
McGill University

Arithmetic. By Fred Marer, Samuel Skolnik, Orda E. Lewis. Little, Brown, Boston, 1960. viii+246 pp.

According to the preface, "This material was first organized for a course initiated at the request of many departments, and of students who felt the lack of proficiency in the fundamentals of arithmetic." (One can only infer that these students are at Los Angeles City College, the institution of two of the authors.)

In this uninspiring book no attempt is made to give the reader a concept of the beauty and pattern of the real number system, or to present any basic principles of operation which eliminate the many special cases. Instead, the first seventeen chapters are filled with the traditional "bag-of-tricks" rules for adding, subtracting, multiplying, and dividing whole numbers and fractions (common, improper, and decimal). Other brief chapters introduce the metric system, proportion, mensuration, and square root. The exercises are prolific but exceedingly simple and unchallenging. Several of the many rules are poorly worded and misleading; for example, to change to percent a fraction whose denominator is 100, one is instructed to "replace the denominator of 100 by the percent sign, %."

It is the opinion of this reviewer that a remedial arithmetic course based on a book of this caliber has no place in an institution of higher education.

VIOLET HACHMEISTER LARNEY
State University of New York,
College of Education, Albany

Mathematical Methods for Digital Computers. Edited by A. Ralston and H. S. Wilf. Wiley, New York, 1960. xi+293 pp. $9.00.

This volume is composed of six main parts: (1) Generation of Elementary Functions, (2) Matrices and Linear Equations, (3) Ordinary Differential Equations, (4) Partial Differential Equations, (5) Statistics, (6) Miscellaneous Methods. Each part is divided into one or more chapters, each of which is an independent entity written by someone who is quite familiar with the subject and who is well known to both the mathematical and the digital computing fraternities. With the exception of the first chapter, the chapters are organized in the same manner, namely: (1) Problem Description, (2) Mathematical Discussion, (3) Calculational Procedure, (4) Flow Chart, (5) Description of Flow Chart, (6) Sample Problem, (7) Subroutines, (8) Memory Requirements, (9) Estimation of Running Time, and (10) References.

This reviewer's experience with beginning graduate students has indicated the efficiency of the above format in that the students were able to program and run problems on a computer with very little assistance using only the material in this book. The editors are to be commended for resisting what must have been a great temptation—using ALGOL for the descriptions!

This is a book that should be in the hands of everyone who intends to do any serious work on a digital computer.

FREDERICK WAY, III
Case Institute of Technology

Analytic Geometry and Calculus. By Abraham Schwartz. Holt, Rinehart and Winston, New York, 1960. xi+864 pp. $9.50.

The first goal set by the author in writing this book was that "the student must be able to read his textbook and learn from it." The liberal use of examples and numerous remarks combined with a reasonably precise presentation are noticeable features of the book which support this goal. Examples are used both to motivate and to illustrate the concepts under discussion. The remarks warn of possible mistakes, help clarify subtle points, or explicitly include other observations which are usually only added by the teacher but might sometimes be overlooked.

The student is assumed to have completed trigonometry and such topics from algebra as mathematical induction, binomial theorem, and elementary theory of equations. Material on inequalities and absolute values is included as an appendix. The study of analytic geometry is delayed until after extensive chapters on differentiation and integration. Some calculus can then be used to study conic sections and curve sketching. Vectors are also used to study analytic geometry. In fact, an important aspect of the book is its thorough use of vector analysis.

The book does not define and prove theorems about limits and continuity until Chapter 7 (p. 375). In this way, the student has a chance to become

familiar with the concepts in an intuitive way and to develop confidence and appreciation before examining the underlying theory. When unable to give a rigorous proof effectively, the author at least gives a plausible argument and observes why it is inadequate. Often, each hypothesis in a theorem is discussed to show the necessity for such a condition. Thus the student is gradually introduced to mathematical sophistication in an informal way.

In the opinion of the reviewer, this text will have wide acceptance by both teachers and students.

JAMES H. MCKAY
Michigan State University Oakland

Intermediate Algebra. By Roy Dubisch, Vernon E. Howes and Steven J. Bryant. Wiley, New York, 1960. xii+286 pp. $4.50.

Intermediate algebra has traditionally emphasized the teaching of techniques for solving algebraic equations and transforming various algebraic expressions. In recent years it has been urged in many quarters that more attention should be paid to algebraic structure, that is to say to the development of algebra as an abstract system from a basic set of axioms. The book under review offers a reasonable compromise between these two points of view. Algebraic techniques and manipulations constitute the authors' primary objective. On the other hand the axiomatic structure of algebra and the role of the axioms in the proof of algebraic theorems has been neither completely neglected nor unduly stressed. The book is written as an introduction to mathematics rather than as a collection of tricks for solving problems. The introduction of notations usually met only in more advanced mathematics will prepare the student for their use in later work. In this category are the use of the summation sign in connection with the progressions, double subscript notation in the discussion of determinants and two by two matrices in connection with complex numbers. A function is carefully defined and the terms domain and range introduced. The chapter on exponents and logarithms stresses the exponential and logarithmic functions and their properties rather than logarithmic computation. The latter is not omitted however. References for further study, a short table of four-place logarithms and answers to odd-numbered problems are included.

D. C. MURDOCH
University of British Columbia

Digital Computer Principles. By Wayne C. Irwin. Van Nostrand, Princeton, N. J., 1960. vi+321 pp. Trade $8.00, Text $6.75.

Well-written, easy-to-understand description of fundamentals upon which modern computers are designed. Aiming at the beginner, the author does a good job of introducing the arithmetic and logic which underlay design. He then shows how the logical functions are mechanized, how information is stored and how arithmetic is carried out. He next treats control, input-output equipment, and programming. Concluding with a discussion of present hardware trends, he

has given us a well-rounded introduction into what is commonly called the hardware concepts of digital computers.

For the potential user, the mathematician or programmer, this book would not be a good primer. However, it can serve effectively to round out the background of the user who has mastered the application of computers and is now motivated to learn more about the how and why of the hardware.

M. A. SHADER
IBM Corporation

Probability, an Introduction. By Samuel Goldberg. Prentice-Hall, Englewood Cliffs, N. J., 1960. ix+322 pp. $5.95.

Few tasks in teaching mathematics are more demanding than that of teaching a mathematically sound course in probability to students with a background of only two years of high school algebra. Such a course necessarily must be the combination of two courses: one on set theory and one on probability. The teacher of this course must indeed rejoice that with this fine text a valuable and much needed aid has become available.

Throughout the book the author shows a high degree of awareness of the difficulties commonly encountered by beginning students of probability. He skillfully makes use of all conceivable devices to facilitate understanding. These include: in the first chapter on sets, each definition is illustrated by numerous examples, many of which point to a later application to probability; proofs of theorems in the algebra of sets by means of membership tables are immediately followed by verifications by means of Venn diagrams, thus enabling the student to develop his intuition, which is so necessary at this stage of his development.

Chapter 2 carefully develops probability theory for finite sample spaces. To avoid the difficulties customarily encountered by students when combinatorial problems and probability are introduced almost simultaneously, the discussion of permutations, combinations (although the latter term is not used in the text outside the preface) and the binomial theorem is deferred until Chapter 3, where they are developed in complete detail. It appears somewhat inconsistent, however, that an introduction to mathematical induction and the use of summation signs, both used extensively in the text, are not also included.

In Chapter 4 random variables are defined as functions on sample spaces, and probability distributions, means, standard deviations, joint probability functions, covariance, and correlation are discussed. To keep the text suitable for a semester course, the author has wisely restricted the discussion of examples of probability distributions to that of the binomial case which constitutes the fifth and last chapter of the text. It includes a brief introduction to testing statistical hypotheses and ends with what certainly should whet any student's appetite for further study: the complete solution of a realistic decision problem.

The text includes 360 well-chosen problems, with answers (in many cases

complete solutions) to half of these provided at the end of the book.

Mathematics or statistics departments offering a basic course in probability using the language and notation of sets should find this text ideally suited to their needs.

H. L. ALDER
University of California, Davis

BRIEF MENTION

Basic Theorems in Matrix Theory, National Bureau of Standards, Applied Mathematics Series No. 57. By Marvin Marcus. Order from Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C., 1960. iv+27 pp. 15¢.

This amazingly inexpensive pamphlet is the finest collection of theorems concerning finite matrices that it has ever been your reviewer's privilege to see. No attempt whatsoever is made to give proofs; this is merely a collection of properties, definitions and theorems. Nevertheless, it is a valuable adjunct and, at its low price, should be in the hands of every member of a class in theory of matrices.

Mathematical Methods in the Social Sciences. By Kenneth J. Arrow, Samuel Karlin and Patrick Suppes. Stanford University Press, Stanford, California, 1960. viii+365 pp. $8.50.

The proceedings of the first Stanford symposium on mathematical methods in the social sciences, held in the summer of 1959, contains nine papers on economics, four papers on management science, and ten papers on psychology.

The Solution of Equations in Integers. By A. O. Gelfond, translated from the Russian by Leo F. Boron. P. Noordhoff-Groningen, Holland, 1960. 72 pp. $1.00.

A fine book that a high school student, a college student, and a professor may all enjoy.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Ball State Teachers College: Dr. J. M. Egar, Texas Agricultural and Mechanical College, has been appointed Associate Professor; Dr. Harry Langman, Ohio Northern University, has been appointed Visiting Professor for 1960–61.

Brown University: Professor Iacapo Barsotti, University of Pittsburgh, has been appointed Professor; Professor Emeritus T. H. Hildebrandt, University of Michigan, has been appointed Visiting Professor; Associate Professor Katsami Nomizu, Catholic University, has been appointed Associate Professor; Dr. R. D. M. Accola, Harvard University, has been appointed Assistant Professor; Dr. Leon Greenberg has been promoted to Assistant Professor.

College of William and Mary: Dr. F. W. Weiler, Ohio State University, has been appointed Assistant Professor; Mr. R. D. H. Jones has been appointed Instructor.

Iowa State University: Dr. W. M. Gilbert, Princeton University, has been appointed Associate Professor; Assistant Professor R. K. Meany, Texas Christian University, has been appointed Assistant Professor; Assistant Professor J. S. Rue and Mr. T. J. Robinson, University of North Dakota, have been appointed Instructors; Assistant Professor M. F. Ruchte has been promoted to Associate Professor.

Miami University: Dr. C. E. Capel, Westinghouse Research Laboratory, Pittsburgh, Pennsylvania, has been appointed Professor; Dr. Yukihiro Kodama, Defense Academy, Yokosuka, Japan, has been appointed Lecturer; Mr. R. F. DeMar has been promoted to Assistant Professor.

Montclair State College: Mrs. Hildegard Howden, Temple University, and Mr. Edward Urband, Bloomfield College, have been appointed Assistant Professors; Associate Professor P. C. Clifford has been promoted to Professor; Assistant Professor J. A. Schumaker has been promoted to Associate Professor.

Oklahoma State University: Assistant Professor S. M. Harmon, Fresno State College, has been appointed Associate Professor; Assistant Professor R. A. Hultquist, DePauw University, has been appointed Assistant Professor; Assistant Professor G. A. Haddock, Arkansas State Teachers College, has been appointed Instructor.

Pamona College: Mr. J. E. Vought, University of Michigan, has been appointed Instructor; Assistant Professor K. L. Cooke has been promoted to Associate Professor.

Southern Illinois University: Professor J. M. H. Olmsted, University of Minnesota, has been appointed Professor and Chairman of the Department of Mathematics; Assistant Professors Marian A. Moore, and J. C. Wilson have been promoted to Associate Professors.

Stanford University: Professors Hans Samelson, University of Michigan, and Ralph Phillips, University of California, Los Angeles, and Associate Professor John Myhill, University of California, Berkeley, have been appointed Professors; Dr. Donald Ornstein, University of Wisconsin, has been appointed Assistant Professor; Dr. P. L. Duren, Massachusetts Institute of Technology, has been appointed Instructor; Assistant Professor James McGregor has been promoted to Associate Professor; Dr. J. W. Lamperti has been promoted to Assistant Professor.

Tennessee Polytechnic Institute: Professor C. G. Phipps, University of Florida, and Associate Professor W. A. Small, Grinnell College, have been appointed Professors; Major General R. C. Hood, Jr., Duke University, has been appointed Associate Professor.

Wayne State University: Professors Ky Fan, University of Notre Dame, and Seymour Sherman, University of Pennsylvania, have been appointed Professors; Associate Professor Melvin Henriksen, Purdue University, has been appointed Visiting Professor; Associate Professor Felix Haas has been appointed Acting Head of the Department of Mathematics; Associate Professor Samuel Kaplan has been promoted to Professor; Assistant Professor Samuel Goldberg has been promoted to Associate Professor.

University of Missouri: Drs. M. D. George, University of Maryland, and L. J. Lange, Bureau of Standards, Boulder, Colorado, have been appointed Assistant Professors; Professor M. V. SubbaRao, Sri Venkateswara University, Tirupati, India, has been appointed Visiting Professor.

University of Tulsa: Dr. J. F. Evans, Pan American Petroleum Corporation, has been appointed Assistant Professor; Mr. R. R. Kinkade, University of Kansas, and Mr. R. M. McDonald, Douglas Aircraft Company, Tulsa, Oklahoma, have been appointed Instructors.

University of Vermont: Assistant Professor D. E. Moser, University of Massachusetts, has been appointed Associate Professor; Assistant Professor J. A. Izzo has been promoted to Associate Professor.

Brother Thomas Warren, San Joaquin Memorial High School, Fresno, California, has been appointed Chairman of the Department of Mathematics at Cathedral High School, Los Angeles, California.

Mr. W. F. Coulson, University of Minnesota, has been appointed Assistant Professor at the University of Alberta.

Mr. R. E. Cummings, Navy Mine Defense Laboratory, Panama City, Florida, has accepted a position as Mathematician with George C. Marshall Space Flight Center, National Aeronautics and Space Administration, Huntsville, Alabama.

Mr. W. R. Derrick, Oklahoma State University, has accepted a position as Mathematician Programmer with International Business Machines, Endicott, New York.

Associate Professor J. N. Eastham, Cooper Union, has been promoted to Professor.

Mr. Eugene Enrione, University of Miami, has accepted a position as Mathematician with Douglas Aircraft, Los Angeles, California.

Professor H. S. Everett, University of Chicago, has been appointed Professor at East Stroudsburg State College.

Associate Professor W. B. Fulks, University of Minnesota, has been appointed Professor at Oregon State College.

Professor Wallace Givens, Wayne State University, has been appointed Professor at Northwestern University.

Mr. Reid Haywood, Martin Company, Baltimore, Maryland, has accepted a position as Senior Program Analyst with the International Electric Corporation, Paramus, New Jersey.

Mr. R. L. Jacobsen, State University of Iowa, has accepted a position as Actuarial Assistant with Northwestern National Life, Minneapolis, Minnesota.

Mr. Walter James, University of Minnesota, has accepted a position with the Minnesota Mining & Manufacturing Company, St. Paul, Minnesota.

Mr. Bernard Levenson, Bureau of Applied Social Research, Columbia University, has been appointed Lecturer and Research Associate at Johns Hopkins University.

Visiting Assistant Professor C. W. Lytle, Drew University, has been appointed Assistant Professor.

Professor W. S. Massey, Brown University, has been appointed Professor at Yale University.

Mr. T. O. McCarley, Notre Dame University, has been appointed Instructor at Central State College, Oklahoma.

Dr. R. A. McHaffey, Rutgers, The State University, has been appointed Assistant Professor at the University of Massachusetts.

Assistant Professor T. D. Oxley, Jr., Kansas State College, has been appointed Assistant Professor at Drake University.

Mr. J. A. Pavelcak, College of St. Thomas, has been appointed Instructor at Merrimack College.

Mr. S. D. Pratico, Fordham University, has accepted a position in the Applied Programming Department of International Business Machines, Manhatten, New York.

Mr. J. R. Stagner, University of Redlands, has accepted a position as Engineering Technician with United Electrodynamics, Pasadena, California.

Assistant Professor J. P. Van Alstyne, Hamilton College, has been promoted to Associate Professor.

Mr. M. L. Whitaker, Florida State University, has been appointed Associate Professor at Radford College.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, New York, has joined General Dynamics Corporation's General Atomic Division in San Diego, California, as Assistant Chairman of the theoretical physics department of the John Jay Hopkins Laboratory for Pure and Applied Science.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE OCTOBER MEETING OF THE INDIANA SECTION

The annual joint meeting of the Indiana Section of the Mathematical Association of America with the Indiana Academy of Science was held on Friday, October 7, 1960, at Manchester College, North Manchester, Indiana. Professor Kermit Carlson of Valparaiso University presided at the morning session and Professor M. E. Shanks of Purdue University at the afternoon session. The meeting was attended by 43 persons, of whom 26 were members of the Association.

Professor J. C. Polley of Wabash College, Chairman of the Indiana School and College Committee on Mathematics, announced a meeting to be held at Purdue University on October 22 under the aegis of the committee. This meeting, the first of five to be sponsored by the committee during the year, had for its theme the preparation of secondary school teachers. Professor E. E. Moise of Harvard University was to be the principal speaker.

Professor R. C. Buck of the University of Wisconsin, Chairman of the Committee for the Undergraduate Program in Mathematics, delivered the invited hour address entitled, "Crises, Past and Present," a commentary on the changes being wrought in the teaching of mathematics and their causes. The following short papers were presented:

1. *Mathematics and the younger generation*, by Professor H. J. Zassenhaus, University of Notre Dame.

2. *The mathematics program at Rose Polytechnic Institute*, by Professor T. P. Palmer, Rose Polytechnic Institute.

3. *The state of mathematics in the State of California*, by Professor Harley Flanders, Purdue University.

4. *Comments on the liaison between high school and college mathematicians*, by Professor A. E. Hallerberg, Valparaiso University.

5. *Interdepartmental seminar: a new course at DePauw University*, by Professor R. J. Thomas, DePauw University.

P. T. MIELKE, *Secretary*

THE OCTOBER MEETING OF THE IOWA SECTION

A combined meeting of the Iowa Section of the Mathematical Association of America and the National Council of Teachers of Mathematics was held October 10 and 11, 1960, at the State University of Iowa, Iowa City, Iowa. The meeting was a Regional Orientation Conference in Mathematics, sponsored by the National Council of Teachers of Mathematics under the Regional Director, Dr. H. V. Price, State University of Iowa. Some twenty members of the Iowa Section attended the conference.

The following program was presented:

1. *Progress in mathematics and its implications for the secondary school*, by Dr. G. B. Price, Executive Secretary, Conference Board of the Mathematical Sciences.

Changes in presenting mathematical materials today were described as being so great as to be called revolutionary. Three causes of this revolution were: (1) advances made in mathematics as a result of mathematical research; (2) automation revolution-introduction of machines that control machines; (3) introduction of large scale automatic computing machines. Implications for schools were: (1) small high schools cannot normally provide the program and teachers needed; (2) many high school teachers need more training; (3) higher standards should be required of teach-

ers of mathematics—from elementary teachers on up; (4) teachers must re-examine techniques they are now using.

2. *Projects in school mathematics*, by Dr. Kenneth Brown, Specialist in Mathematics, United States Office of Education.

Dr. Brown discussed the school mathematics projects that are underway in various parts of the country. He pointed out that these projects are discussed briefly in "Aids for Mathematics Education: Mathematics—A Universal Language of Modern Civilization: OE-29013," U. S. Department of Health, Education and Welfare, Office of Education, Washington 25, D. C.

3. *University of Illinois program* (*UICSM*), by Miss Grace Wandke, Barrington Consolidated High School, Barrington, Illinois.

4. *University of Maryland program* (*UMMaP*), by Mrs. Ruth Brown, McKistry Junior High School, Waterloo, Iowa.

5. *Ball State program*, by Mr. Wilson Banks, Pleasant Valley-Riverdale High School, Bettendorf, Iowa.

6. *School Mathematics Study Group program*, by Miss Joan Tanzer, Mechanical Arts High School, St. Paul, Minnesota.

7. *Implementing a new mathematics program for secondary schools*, by Dr. W. E. Ferguson, Newton High School, Newtonville, Massachusetts.

Eight basic steps were listed for implementing a new mathematics program: (1) recognition by the school authorities of the need for a new mathematics program; (2) adequate preparation of teachers in the mathematics that is now being taught for the first time in secondary schools; (3) selection of a new program; (4) selection of students for the program; (5) informing parents about the new program; (6) informing other members of the school system about the new program and its implications for the mathematics program, Kindergarten through Grade 12; (7) continuation of teacher preparation for carrying the new program to higher grades and also lower grades; (8) provision for adequate time and compensation for carrying on the new program year after year.

8. *Question and answer*, by a panel consisting of Dr. G. B. Price, Dr. Kenneth Brown, Dr. W. E. Ferguson, and Mr. F. B. Allen, Lyons Township High School and Junior College, La Grange, Illinois.

E. L. CANFIELD, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University on October 28 and 29, 1960. Professor Kathrine C. Mires, Chairman of the Section, presided. There were 160 persons in attendance, including 71 members of the Association.

The following officers were elected for a one-year term: Chairman, Professor J. A. Nickel, Oklahoma City University; Vice-Chairman, Professor R. R. Murphy, Panhandle A and M College; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association and contains expository papers believed to be of particular interest to high school teachers. Research papers are presented at the spring meeting. The following papers were presented:

1. *Time delays in waiting for a traffic light*, by Professor J. A. Nickel, Oklahoma City University.

2. *Completeness in uniform spaces*, by Professor J. C. Mathews, University of Oklahoma.

3. *Extremal elements of convex cones*, by Professor E. K. McLachlan, Oklahoma State University.

4. *Mathematical faces of flexagons*, by Professor C. O. Oakley, Haverford College.

5. *Sums of series*, by Professor W. A. Rutledge, University of Tulsa.

6. *A commentary on normal forms in sentential calculus*, by Mr. L. E. DeNoya, Oklahoma State University.

7. *SMSG mathematics in Junior High*, luncheon address by Miss Veryl Shult, Washington, D. C.

8. *General aims and purposes of the CUPM recommendations for teacher training in mathematics*, by Professor Robert Wisner, Executive Director, Committee on the Undergraduate Program of the Mathematical Association of America.

Professor J. H. Zant of the Oklahoma Committee for the Improvement of Mathematics Instruction, and Mr. F. R. Born of the State Office of Education, were asked to comment on these recommendations. Each college giving teacher training was invited to have an official representative present at this meeting. Dr. Wisner's talks were follows by open discussion and questions from the floor.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-second Summer Meeting, Oklahoma State University, Stillwater, Oklahoma, August 28–30, 1961.

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 6, 1961.

ILLINOIS, University of Illinois, Urbana, May 12–13, 1961.

INDIANA, Rose Polytechnic Institute, Terre Haute, May 6, 1961.

IOWA, Simpson College, Indianola, April 14, 1961.

KANSAS, Ottawa University, April 15, 1961.

KENTUCKY, Western Kentucky State College, Bowling Green, Spring, 1961.

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METROPOLITAN NEW YORK, Fordham University, New York, April 15, 1961

MICHIGAN, Wayne State University, Detroit, March 25, 1961.

MINNESOTA

MISSOURI, University of Missouri, Columbia, April 22, 1961.

NEBRASKA, University of Nebraska, Lincoln, April 15, 1961.

NEW JERSEY

NORTHEASTERN, University of Vermont, Burlington, June 20, 1961.

NORTHERN CALIFORNIA

OHIO, Ohio Wesleyan University, Delaware, May 6, 1961.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 17, 1961.

PHILADELPHIA, Ursinus College, Collegeville, Pennsylvania, November 25, 1961.

ROCKY MOUNTAIN, University of Colorado, Boulder, April 28–29, 1961.

SOUTHEASTERN, Wofford College, Spartanburg, South Carolina, April 7–8, 1961.

SOUTHERN CALIFORNIA, University of California, Santa Barbara, March 11, 1961.

SOUTHWESTERN, University of Arizona, Tucson, March 17–18, 1961.

TEXAS, Stephen F. Austin State College, Nacogdoches, April 14–15, 1961.

UPPER NEW YORK STATE, Harpur College, Binghamton, April 29, 1961.

WISCONSIN, University of Wisconsin, Madison, May 13, 1961.

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